

Exponential Stability and Initial Value Problems for Evolutionary Equations

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Nomenclature

Sets

$\mathbb{R}_{\geq a}$	the set of reals greater/less than or equal to $a \in \mathbb{R}$.
$\mathbb{C}_{\operatorname{Re} \geq \alpha}$	the set of all complex numbers with real part greater than, equal to or less than some real number α .
$B(x, r)$	the open ball with center x and radius r .
$B[x, r]$	the closed ball with center x and radius r .

Spaces

$C(\mathbb{R}; H)$	the space of continuous functions $f : \mathbb{R} \rightarrow H$.
$C^k(\mathbb{R}; H)$	the space of k times continuously differentiable functions $f : \mathbb{R} \rightarrow H$.
$C^\infty(\mathbb{R}; H)$	$\bigcap_{k \in \mathbb{N}} C^k(\mathbb{R}; H)$.
$C_c(\mathbb{R}; H)$	the space of continuous functions $f : \mathbb{R} \rightarrow H$ having compact support.
$C_c^\infty(\mathbb{R}; H)$	$C^\infty(\mathbb{R}; H) \cap C_c(\mathbb{R}; H)$.
$\mathcal{S}(\mathbb{R}; H)$	the Schwartz-space of rapidly decreasing functions $f : \mathbb{R} \rightarrow H$.
$\mathcal{H}^2(\mathbb{C}_{\operatorname{Re} > a})$	the Hardy space of analytic functions f on the half plane $\mathbb{C}_{\operatorname{Re} > a}$ with $\sup_{\varrho > a} \int_{\mathbb{R}} f(it + \varrho) ^2 dt < \infty$.
$C_\varrho(\mathbb{R}; H)$	space of continuous functions $f : \mathbb{R} \rightarrow H$ with $\sup_{t \in \mathbb{R}} f(t) e^{-\varrho t} _H < \infty$ for some $\varrho \in \mathbb{R}$.
$H_\varrho(\mathbb{R}; H)$	the Hilbert space of all measurable functions $f : \mathbb{R} \rightarrow H$ satisfying $\int_{\mathbb{R}} f(t) ^2 \exp(-2\varrho t) dt < \infty$.
$H_\varrho^1(\mathbb{R}; H)$	the domain of $\partial_{0,\varrho}$ equipped with the norm $ \partial_{0,\varrho} \cdot _{H_\varrho(\mathbb{R}; H)}$.
$H_\varrho^{-1}(\mathbb{R}; H)$	the first extrapolation space of $\partial_{0,\varrho}$, the completion of $H_\varrho(\mathbb{R}; H)$ with respect to the norm $ \partial_{0,\varrho}^{-1} \cdot _{H_\varrho(\mathbb{R}; H)}$.
$L(X; Y)$	the space of bounded linear operators $T : X \rightarrow Y$ between two Banach spaces X and Y .

Nomenclature

$L(X)$	the space $L(X; X)$.
X'	the dual space of a space X .
$\text{His}_\varrho(M, A)$	the space of admissible histories for a material law M and an operator A in $H_\varrho(\mathbb{R}; H)$.
$\text{IV}_\varrho(M, A)$	the space of admissible initial values for a material law M and an operator A in $H_\varrho(\mathbb{R}; H)$, the space $\{g(0-); g \in \text{His}_\varrho(M, A)\}$.
D_ϱ	the space $\{(g(0-), g); g \in \text{His}_\varrho(M, A)\}$.
X_ϱ^μ	the closure of D_ϱ in $H \times H_\mu(\mathbb{R}_{\leq 0}; H)$, the natural domain of $T_{(M, A)}$.

Operators

grad_0	the gradient on $L_2(\Omega)$ with homogeneous Dirichlet boundary values.
grad	the gradient on $L_2(\Omega)$.
div_0	the divergence on $L_2(\Omega)^n$ with homogeneous Neumann boundary values.
div	the divergence on $L_2(\Omega)^n$.
curl_0	the rotation on $L_2(\Omega)^3$ with homogeneous electrical boundary condition.
curl	the rotation on $L_2(\Omega)^3$.
Grad_0	the symmetrized Jacobian on $L_2(\Omega)^n$ with homogeneous Dirichlet boundary values.
Grad	the symmetrized Jacobian on $L_2(\Omega)^n$.
Div_0	the row-wise divergence on $L_2(\Omega)^{n \times n}$ with homogeneous Neumann boundary values.
Div	the row-wise divergence on $L_2(\Omega)^{n \times n}$.
\mathcal{F}	the Fourier transformation.
\mathcal{L}_ϱ	the Fourier-Laplace transformation.
$\partial_{0, \varrho}$	the derivative on $H_\varrho(\mathbb{R}; H)$, the time-derivative.
$\varrho(T)$	the resolvent set of an operator T .
$\sigma(T)$	the spectrum of an operator T .
$C\sigma(T)$	the continuous spectrum of an operator T .
A^*	the adjoint of an operator A .

Nomenclature

τ_h	the translation operator given by $\tau_h f = f(\cdot + h)$.
$k*$	the convolution operator with a kernel k .
S_ϱ	the solution operator $\overline{(\partial_{0,\varrho} M(\partial_{0,\varrho}) + A)^{-1}}$.

Further Symbols

$\langle \cdot \cdot \rangle_H$	inner product on H , linear in the second and conjugate linear in the first argument.
z^*	the conjugate of a complex number z .
$ \cdot _X$	norm on X .
$\ \cdot\ $	the operator norm on a space $L(X; Y)$.
$\lambda(A)$	the Lebesgue measure of a Borel set A .
\hookrightarrow	continuous embedding.
\widehat{f}	the Fourier transform of a function f .
$\text{spt } f$	the support of a function f .
$T_{(M,A)}$	the C_0 -semigroup associated with a material law M and an operator A .
$\omega(T)$	the growth type of a C_0 -semigroup T .
$s_0(M, A)$	the abscissa of boundedness of a material law M and an operator A .

Introduction

In this thesis we deal with so-called *linear evolutionary equations*, a class of partial differential equations which was introduced by Picard in 2009, [Pic09]. These equations can be written as

$$\partial_t w + Au = f,$$

where A is a densely defined closed linear operator on a Hilbert space H , serving as the state space, and ∂_t denotes the time derivative. The right-hand side f is given and w and u are to be determined. Of course, in order to obtain a solvable problem, we need to link the unknowns w and u . This is done by a so-called *material law* \mathcal{M} , a suitable linear operator acting in space-time. The relation is then given as

$$w = \mathcal{M}u$$

and thus, eliminating w from our abstract differential equation via the material relation, we arrive at a problem of the form

$$(\partial_t \mathcal{M} + A)u = f, \tag{0.1}$$

where now u is the unknown of interest. Clearly, w can be re-constructed from u via the material relation.

Well-posedness in the sense of Hadamard of problems of the form (0.1) is our first topic of interest. This encompasses existence and uniqueness of a solution u and its continuous dependence on the data f . For attack this question, we need to first define, what we mean by a solution of (0.1). In the case $\mathcal{M} = 1$, the well established theory semigroups could be applied. There it is common to write the problem as an ordinary differential equation $\partial_t u = -Au + f$ in an infinite dimensional state space and to define so-called *mild solutions* u . These are continuous functions satisfying the equation in an integrated sense. The existence and uniqueness of mild solutions is then equivalent to the existence of a C_0 -semigroup generated by $-A$ (for the theory of C_0 -semigroups we refer to the monographs [Eng98, ABHN11, Paz83]).

Here, however, we follow a different approach with the aim to obtain well-posedness of equations of the general form (0.1). The main idea is to establish the operator sum $\partial_t \mathcal{M} + A$ as a suitable operator acting in space-time. Then, the uniqueness and existence of solutions is equivalent to the bijectivity of this operator and to obtain continuous dependence of u on the data requires for the continuity of the inverse operator $(\partial_t \mathcal{M} + A)^{-1}$. In order to establish the operator sum properly, we need to define the underlying function space and the operators ∂_t and \mathcal{M} . As the underlying space we will use the Hilbert space $H_\varrho(\mathbb{R}; H)$ consisting of (equivalence classes of) measurable functions $f : \mathbb{R} \rightarrow H$, which are square integrable with respect to the exponentially weighted Lebesgue measure $e^{-2\varrho t} dt$ with $\varrho \in \mathbb{R}$, equipped with its canonical inner product. Then, the temporal derivative ∂_t can be defined as the closure of the derivative of test functions $C_c^\infty(\mathbb{R}; H)$. The so obtained operator, which will be denoted by $\partial_{0,\varrho}$, turns out to be a normal operator which is continuously invertible if and only if $\varrho \neq 0$.

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The idea to define the derivative in that way goes back to [Pic89]. As a normal operator, $\partial_{0,\varrho}$ has a spectral representation, which is given by the so-called Fourier-Laplace transformation. Using this spectral representation, we will define \mathcal{M} to be an operator-valued function of $\partial_{0,\varrho}$ denoted by $M(\partial_{0,\varrho})$. Hence, (0.1) takes the form

$$(\partial_{0,\varrho}M(\partial_{0,\varrho}) + A)u = f$$

on $H_\varrho(\mathbb{R}; H)$ and well-posedness of this problem can be reformulated as the continuous invertibility of $\partial_{0,\varrho}M(\partial_{0,\varrho}) + A$ on $H_\varrho(\mathbb{R}; H)$. We emphasize that the so obtained solutions are not continuous in general and the so introduced notion of solutions is weaker than the notion of mild solutions in the theory of C_0 -semigroups.

The above mentioned perspective of looking at differential equations as operator equations involving a sum of two unbounded operators, is by no means new and was indeed successfully employed earlier and also in the more general setting of Banach space theory. In particular, when dealing for example with the issue of so-called maximal regularity of evolution equations, this perspective appears quite natural. We refer here to the seminal paper of da Prato and Grisvard [dPG75], where the closedness, closability and continuous invertibility of operator sums are studied in a general Banach space setting. Since we are, however, restricting ourselves to the Hilbert space setting and since the operators involved have the above mentioned special structure, these properties can be obtained easily and we do not need to employ the deep and intricate results of [dPG75]. A further reference is [BH76], where the authors derive abstract results for the range of the sum of two binary relations and apply them to derive existence results for differential equations and inclusions. In [FY99] an abstract operator equation of the form $(TM - L)u = f$ is considered, and conditions on the operators involved are derived to obtain closability and bounded invertibility of this operator sum. In particular, setting $T = \partial_{0,\varrho}$, $M = M(\partial_{0,\varrho})$ and $L = -A$, we end up with an evolutionary equation. However, in general these operators do not satisfy the constraints imposed on the operators T, M and L in [FY99].

Beyond the well-posedness in the sense of Hadamard, causality, as a distinguishing property of evolutionary problems, needs to be addressed. By causality we mean, roughly speaking, that the solution vanishes as long as the source term does. More precisely, if f is zero on $] - \infty, a]$ for some $a \in \mathbb{R}$, then also u vanishes on the same interval $] - \infty, a]$. This can be seen as a characteristic property of time evolution. The requirement of causality imposes a strong constraint on admissible material laws, namely that the operator-valued function describing the material properties needs to be analytic. This is due to a theorem by Paley and Wiener, which characterizes those $L_2(\mathbb{R})$ -functions which are supported on the positive real axis by properties of their Laplace transform (see [PW34] and Theorem A.7 of this thesis). For a discussion of causality and the concept of causal differential equations we refer to the monograph [LLM10].

As it turns out, the framework of evolutionary equations covers a broad class of linear time-shift invariant (i.e. autonomous) partial differential equations. For instance, most (if not all) of the linear differential equations of classical mathematical physics can be written as evolutionary equations in our sense. In particular, coupling phenomena can be conveniently incorporated and analyzed (see e.g. [MPTW16b, MPTW16a, MPTW15, MP10, PTW15b]). Moreover, other types of differential equations can be written as evolutionary problems such as integro-differential equations (see [Tro15c]), fractional differential equations (see [PTW15a])

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or delay differential equations (see [KPS⁺14]). Also problems with transmission conditions and complicated boundary conditions, such as impedance boundary conditions, can be treated within this framework (see [PSTW16, Tro14a]). We also mention that certain generalizations of evolutionary problems were also already investigated. For instance, replacing the material law operator $M(\partial_{0,\varrho})$ by more general operators, which fail to commute with the derivative $\partial_{0,\varrho}$, allows for the treatment of certain non-autonomous problems (see [PTWW13, Wau15b]). Another direction of generalization is to incorporate nonlinear terms by replacing the linear operator A by a nonlinear operator, or even more general, by a binary relation. The problem class obtained is referred to as evolutionary inclusions (see [Tro12, Tro13a, TW14]). A further direction in current research employing the framework of evolutionary problems is the homogenization of partial differential equations, or, to put it in another perspective, the continuous dependence of the solution u on the coefficient operators involved (see [Wau12, Wau13, Wau14b, Wau14a, Wau16b] and in particular the habilitation thesis [Wau16a]).

Having secured well-posedness of an evolutionary problem, it is natural to investigate qualitative properties of its solution u . As a first property of interest of such a solution u , we mention its long-time asymptotics. More precisely, we ask whether the solution decays to zero with an exponential rate, provided that the given right-hand side f does. This property is known as *exponential stability* and was studied in [Tro13b, Tro14b, Tro15a]. The first difficulty is that the solutions of evolutionary problems are not continuous, so a classical point-wise estimate cannot be used. The main idea to overcome this is to relax the notion of exponential stability by the constraint that the solution should be square integrable with respect to $e^{2\mu t} dt$ for some $\mu > 0$, that is $u \in H_{-\mu}(\mathbb{R}; H)$. Hence, the question of exponential stability reduces to the question whether the solution operator $(\partial_{0,\varrho}M(\partial_{0,\varrho}) + A)^{-1}$, which is independent of the parameter ϱ for large enough ϱ , can be extended to $H_{-\mu}(\mathbb{R}; H)$ for some positive μ . Similar ideas were already used in the study of exponential stability for evolution equations via semigroups, where the exponential stability can be defined in the point-wise sense. As it turns out, at least in the Hilbert space setting, the extension of the solution operator indeed yields the exponential decay of solutions in the point-wise sense by the famous Gearhart-Prüß Theorem (see [Prü84, Gea78, Her83]).

It should be mentioned that in the framework of semigroups a number of other stability results are well-known. There is for example the celebrated Arendt-Batty-Lyubich-Vũ Theorem [AB88, LV88], giving a criterion for the strong convergence of a C_0 -semigroup to 0, a recent result by Borichev and Tomilov [BT10], providing a characterization of polynomial decay of solutions of evolution equations in Hilbert spaces, or the Theorem of Batty-Duyckaerts [BD08] for further decay estimates of C_0 -semigroups, which are obtained by Laplace transform techniques.

We recall that evolutionary equations, i.e. equations of the form (0.1), are operator equations on $H_\varrho(\mathbb{R}; H)$. Hence, the functions involved are supported on the whole real line in general. However, in the classical theory for evolution equations it is common to consider functions supported on the positive real line and to add an initial condition for the unknown u . Such initial value problems can also be incorporated into the framework of evolutionary problems. First, we note that due to the causality of the solution operator, right-hand sides f supported on $\mathbb{R}_{\geq 0}$ also yield solutions supported on $\mathbb{R}_{\geq 0}$. So, the question is how to incorporate an initial condition for u . Following the ideas developed in [PM11], this can be done by adding suitable distributional source terms on the right-hand side of (0.1). More precisely, an initial condition turns out to be encoded as an impulse at time zero by a Dirac-delta distribution at the initial

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time zero. Indeed, employing a Sobolev embedding theorem and the causality of the solution operator, one can show that for this modified right-hand side the solution u indeed attains the initial value. However, in order to do so, one needs to extend the solution operator to distributional right-hand sides. This can be done easily, using the extrapolation spaces for the time derivative operator $\partial_{0,\varrho}$. For the theory of extrapolation spaces and their application to partial differential equations we refer to [PM11, Pic00] and to [Nag97] with a focus on C_0 -semigroups.

Since evolutionary equations also cover purely algebraic equations (choose for a simple example $\mathcal{M} = \partial_{0,\varrho}^{-1}$) or, more generally, differential-algebraic equations, an initial condition for the unknown u does not make sense for every choice of \mathcal{M} . Thus, one has to address the question what meaningful initial values are in presence of more complicated material laws. The discussion of admissible spaces of initial conditions is well-known in the theory of differential-algebraic equations, see e.g. [KM06, Rei07]. Moreover, since also certain classes of delay equations are covered by our approach, where it seems to be more natural to prescribe whole histories instead of initial values at zero, the question arises how those given histories could be incorporated as data for evolutionary equations.

Once initial value problems for evolutionary equations are settled, one can go further and ask for more regular solutions, or, more specifically, when and where solutions are continuous in time. This will be done by associating a C_0 -semigroup to the given initial value problem, which of course requires some additional constraints on the operators involved. Having the C_0 -semigroup at hand, it is then possible to compare the exponential stability results for evolutionary equations with the classical exponential stability for the associated semigroup and to provide a generalization of the Gearhart-Prüß Theorem.

This thesis has three chapters and an appendix. In the first chapter we introduce the framework of evolutionary equations and study their well-posedness. This includes the classical Hadamard requirements of uniqueness, existence and continuous dependence of solutions as well as the causality of the solution operator. In Section 1.3 we discuss several examples of partial differential equations and show that these problems can be written as evolutionary equations and their well-posedness can be shown using the results provided in Chapter 1. These examples cover classical linear partial differential equations from mathematical physics as well as classes of delay equations and integro-differential equations. The second chapter is devoted to the exponential stability of evolutionary problems. There, we introduce the notion of exponential stability and derive criteria for the operators involved to obtain exponentially stable evolutionary problems. In particular, we are focusing on second-order problems, where we provide a way to reformulate the second order problem as a suitable first order problem, for which the exponential stability can be shown. Again, at the end of Chapter 2, we discuss several examples of different classes of partial differential equations, whose asymptotics are studied with the help of the abstract results of the previous sections. Finally, in Chapter 3 we address the problem of admissible initial values and histories for a given evolutionary problem and discuss how these given data could be incorporated into the setting of evolutionary problems. For doing so, we introduce the extrapolation space $H_{\varrho}^{-1}(\mathbb{R}; H)$ and generalized cut-off operators on that space. These operators will then be used to formulate initial value problems and problems with prescribed histories in Section 3.2. Section 3.3 is devoted to the regularity of pure initial value problems and provides a Hille-Yosida type result for the existence of an associated C_0 -semigroup on a suitable Hilbert space. Moreover, we present a generalization of the Gearhart-Prüß Theorem to a class of evolutionary problems. In the last section in

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Chapter 3 we discuss several examples to illustrate the results. In the appendix we recall some well-known theorems, which will be needed in particular in Chapter 3.

We assume that the reader is familiar with basic functional analysis, in particular with Hilbert space theory. For these topics we refer to the monographs [Yos95, Wei80, Phi59, Rud91]. Throughout, all Hilbert spaces are assumed to be complex and the inner product $\langle \cdot | \cdot \rangle$ is assumed to be linear in the second and conjugate linear in the first argument.

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- Warning: May contain nuts¹ -

¹This warning is stated for the case, that this thesis will be distributed in Ankh-Morpork, see [PSC02].

1. The framework of evolutionary problems

In this chapter we introduce the framework of evolutionary equations. The class was introduced by R. Picard in [Pic09] and it turned out that it covers a broad class of different types of differential equations. The key feature of the framework presented here is the realization of the time-derivative as an accretive operator on a suitable Hilbert space, which can be seen as an L_2 -analogue of the continuous functions equipped with the Morgenstern-norm for the treatment of ordinary differential equations (see [Mor52]). The idea of introducing the time derivative in an exponentially weighted L_2 -space originates from [Pic89], where certain integral transformations on Hilbert spaces were considered as functions of suitable differential operators. We begin by introducing this Hilbert space and the time derivative established on it. After that, we define what we mean by an evolutionary problem and address the question of well-posedness and causality for this class of problems. In the last section of this chapter we present some examples.

1.1. The Hilbert space setting and the time derivative

Throughout this section let H be a Hilbert space.

Definition. Let $\varrho \in \mathbb{R}$. We define

$$\mathcal{L}_{2,\varrho}(\mathbb{R}; H) := \left\{ f : \mathbb{R} \rightarrow H ; f \text{ measurable, } \int_{\mathbb{R}} |f(t)|_H^2 e^{-2\varrho t} dt < \infty \right\}.$$

Moreover, we introduce the equivalence relation \cong on $\mathcal{L}_{2,\varrho}(\mathbb{R}; H)$ by $f \cong g$ if $f(t) = g(t)$ for a.e. $t \in \mathbb{R}$ with respect to the Lebesgue measure on \mathbb{R} . As usual we define

$$H_\varrho(\mathbb{R}; H) := \mathcal{L}_{2,\varrho}(\mathbb{R}; H) / \cong$$

which is a Hilbert space with respect to the inner product

$$\langle f | g \rangle_\varrho := \int_{\mathbb{R}} \langle f(t) | g(t) \rangle_H e^{-2\varrho t} dt \quad (f, g \in H_\varrho(\mathbb{R}; H)).$$

Remark 1.1.1.

- (a) If we choose $\varrho = 0$ in the latter definition, $H_0(\mathbb{R}; H)$ coincides with the usual Bochner space $L_2(\mathbb{R}; H)$.

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(b) For $\varrho \in \mathbb{R}$ the operator

$$\begin{aligned} e^{-\varrho m} : H_\varrho(\mathbb{R}; H) &\rightarrow L_2(\mathbb{R}; H) \\ f &\mapsto (t \mapsto e^{-\varrho t} f(t)) \end{aligned}$$

is obviously unitary. In particular, since $e^{-\varrho m}$ is a bijection on $C_c^\infty(\mathbb{R}; H)$, we derive that $C_c^\infty(\mathbb{R}; H)$ is dense in $H_\varrho(\mathbb{R}; H)$.

In fact, we can also show a slightly stronger result than the density of $C_c^\infty(\mathbb{R}; H)$ in $H_\varrho(\mathbb{R}; H)$.

Lemma 1.1.2. *Let $\varrho, \mu \in \mathbb{R}$ and $f \in H_\varrho(\mathbb{R}; H) \cap H_\mu(\mathbb{R}; H)$. Then there exists a sequence $(\varphi_n)_{n \in \mathbb{N}}$ in $C_c^\infty(\mathbb{R}; H)$ such that $\varphi_n \rightarrow f$ in $H_\varrho(\mathbb{R}; H)$ and $H_\mu(\mathbb{R}; H)$.*

Proof. Without loss of generality let $\varrho < \mu$. For $n \in \mathbb{N}$ we define $f_n(t) := \chi_{[-n, n]}(t)f(t)$ for $t \in \mathbb{R}$ and get $f_n \rightarrow f$ in $H_\varrho(\mathbb{R}; H)$ and $H_\mu(\mathbb{R}; H)$ as $n \rightarrow \infty$ by dominated convergence. For $n \in \mathbb{N}$ we find $\varphi_n \in C_c^\infty(\mathbb{R}; H)$ with $\text{spt } \varphi_n \subseteq [-n, n]$ such that $|\varphi_n - f_n|_\mu < \frac{1}{2n} e^{(\varrho - \mu)n}$. Hence,

$$|\varphi_n - f_n|_\varrho = \left(\int_{-n}^n |\varphi_n(t) - f_n(t)|_H^2 e^{-2\varrho t} dt \right)^{\frac{1}{2}} \leq |\varphi_n - f_n|_\mu e^{(\mu - \varrho)n} < \frac{1}{2n}$$

and thus, we derive

$$\begin{aligned} |\varphi_n - f|_\mu &< \frac{1}{2n} e^{(\varrho - \mu)n} + |f_n - f|_\mu \rightarrow 0, \\ |\varphi_n - f|_\varrho &< \frac{1}{2n} + |f_n - f|_\varrho \rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$. □

We now define the time derivative as an operator acting on $H_\varrho(\mathbb{R}; H)$.

Lemma 1.1.3. *Let $\varrho \in \mathbb{R}$. Then the operator*

$$\begin{aligned} \partial_{0, \varrho, c} : C_c^\infty(\mathbb{R}; H) &\subseteq H_\varrho(\mathbb{R}; H) \rightarrow H_\varrho(\mathbb{R}; H) \\ \varphi &\mapsto \varphi' \end{aligned}$$

is closable.

Proof. Let $\varphi, \psi \in C_c^\infty(\mathbb{R}; H)$. Then we compute, by using integration by parts,

$$\begin{aligned} \langle \partial_{0, \varrho, c} \varphi | \psi \rangle_\varrho &= \int_{\mathbb{R}} \langle \varphi'(t) | \psi(t) \rangle_H e^{-2\varrho t} dt \\ &= - \int_{\mathbb{R}} \langle \varphi(t) | \psi'(t) \rangle_H e^{-2\varrho t} dt + 2\varrho \int_{\mathbb{R}} \langle \varphi(t) | \psi(t) \rangle_H e^{-2\varrho t} dt \\ &= \langle \varphi | -\partial_{0, \varrho, c} \psi + 2\varrho \psi \rangle_\varrho, \end{aligned}$$

which shows $\psi \in D(\partial_{0, \varrho, c}^*)$ with $\partial_{0, \varrho, c}^* \psi = -\partial_{0, \varrho, c} \psi + 2\varrho \psi$. Hence, $\partial_{0, \varrho, c} \subseteq -\partial_{0, \varrho, c}^* + 2\varrho$, which in particular yields the closability of $\partial_{0, \varrho, c}$. □

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Definition. Let $\varrho \in \mathbb{R}$. We define $\partial_{0,\varrho} := \overline{\partial_{0,\varrho,c}}$ and $H_\varrho^1(\mathbb{R}; H) := D(\partial_{0,\varrho})$. We equip $H_\varrho^1(\mathbb{R}; H)$ with the inner product

$$\langle f|g \rangle_{\varrho,1} := \langle f|g \rangle_\varrho + \langle \partial_{0,\varrho} f | \partial_{0,\varrho} g \rangle_\varrho \quad (f, g \in H_\varrho^1(\mathbb{R}; H)).$$

Note that $H_\varrho^1(\mathbb{R}; H)$ is a Hilbert space.

Our next goal is to show that $\partial_{0,\varrho}$ is normal. For doing so, we show that the Fourier-Laplace transformation $\mathcal{L}_\varrho : H_\varrho(\mathbb{R}; H) \rightarrow L_2(\mathbb{R}; H)$ defined as the unitary extension of

$$(\mathcal{L}_\varrho \varphi)(t) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-(i t + \varrho)s} \varphi(s) \, ds \quad (\varphi \in C_c^\infty(\mathbb{R}; H), t \in \mathbb{R})$$

establishes a spectral representation for $\partial_{0,\varrho}$. For a deeper study of the Fourier-Laplace transformation we refer to Appendix A.

Proposition 1.1.4. *Let $\varrho \in \mathbb{R}$. Then we have*

$$\partial_{0,\varrho} = \mathcal{L}_\varrho^* (\mathfrak{m} + \varrho) \mathcal{L}_\varrho,$$

where $\mathfrak{m} : D(\mathfrak{m}) \subseteq L_2(\mathbb{R}; H) \rightarrow L_2(\mathbb{R}; H)$ is defined as multiplication with the argument with maximal domain, i.e. $(\mathfrak{m} f)(t) := t f(t)$ for $t \in \mathbb{R}$ and $f \in D(\mathfrak{m})$ with

$$D(\mathfrak{m}) := \{g \in L_2(\mathbb{R}; H) ; (t \mapsto t g(t)) \in L_2(\mathbb{R}; H)\}.$$

Proof. The proof will be done in three steps.

Firstly, we show that $\mathcal{S}_\varrho(\mathbb{R}; H) := \{f : \mathbb{R} \rightarrow H ; e^{-\varrho m} f \in \mathcal{S}(\mathbb{R}; H)\}$ is a core for $\partial_{0,\varrho}$ and $\partial_{0,\varrho} f = f'$ for $f \in \mathcal{S}_\varrho(\mathbb{R}; H)$. Here, $\mathcal{S}(\mathbb{R}; H)$ denotes the Schwartz-space of rapidly decreasing smooth functions with values in H . By definition of $\partial_{0,\varrho}$, the space $C_c^\infty(\mathbb{R}; H) \subseteq \mathcal{S}_\varrho(\mathbb{R}; H)$ is a core for $\partial_{0,\varrho}$. Thus, it suffices to prove that $\mathcal{S}_\varrho(\mathbb{R}; H) \subseteq D(\partial_{0,\varrho})$. For doing so, let $f \in \mathcal{S}_\varrho(\mathbb{R}; H)$. Moreover, let $\psi \in C_c^\infty(\mathbb{R})$ such that $\psi = 1$ on $[-1, 1]$ and set $\psi_n(t) := \psi(n^{-1}t)$ for $n \in \mathbb{N}, t \in \mathbb{R}$. Then, for each $n \in \mathbb{N}$ we have $\psi_n f \in C_c^\infty(\mathbb{R}; H)$ and $\psi_n f \rightarrow f$ in $H_\varrho(\mathbb{R}; H)$ as $n \rightarrow \infty$ by dominated convergence. Moreover, $\partial_{0,\varrho}(\psi_n f) = \psi'_n f + \psi_n f' \rightarrow f'$ as $n \rightarrow \infty$ in $H_\varrho(\mathbb{R}; H)$, again by dominated convergence. This shows the claim.

Secondly, we show that $C_c^\infty(\mathbb{R}; H)$ is a core for \mathfrak{m} , and thus, also for $\mathfrak{m} + \varrho$. Let $f \in D(\mathfrak{m})$ and set $f_n(t) := \chi_{[-n,n]}(t) f(t)$ for $t \in \mathbb{R}, n \in \mathbb{N}$. Then, clearly $f_n \rightarrow f$ and $\mathfrak{m} f_n \rightarrow \mathfrak{m} f$ in $L_2(\mathbb{R}; H)$ as $n \rightarrow \infty$ by dominated convergence. Hence, for $\varepsilon > 0$ we find $n \in \mathbb{N}$ such that

$$|f - f_n|_{L_2}, |\mathfrak{m} f - \mathfrak{m} f_n|_{L_2} < \varepsilon.$$

Choose now $\varphi \in C_c^\infty(\mathbb{R}; H)$ with $\text{spt } \varphi \subseteq [-n-1, n+1]$ and $|\varphi - f_n|_{L_2} < \frac{\varepsilon}{n+1}$. Then

$$|\mathfrak{m} f_n - \mathfrak{m} \varphi|_{L_2} \leq (n+1) |f_n - \varphi|_{L_2} < \varepsilon$$

and hence,

$$|f - \varphi|_{L_2}, |\mathfrak{m} f - \mathfrak{m} \varphi|_{L_2} < 2\varepsilon,$$

which shows the assertion.

Finally, we now show the asserted statement. Let $f \in \mathcal{S}_\varrho(\mathbb{R}; H)$. Then $\mathcal{L}_\varrho f \in \mathcal{S}(\mathbb{R}; H) \subseteq$

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$D(\mathbf{i}m + \varrho)$ (cp. Proposition A.2) and we compute, using the first step

$$\begin{aligned} (\mathcal{L}_\varrho \partial_{0,\varrho} f)(t) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-(\mathbf{i}t + \varrho)s} f'(s) \, ds \\ &= (\mathbf{i}t + \varrho) \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-(\mathbf{i}t + \varrho)s} f(s) \, ds \\ &= ((\mathbf{i}m + \varrho) \mathcal{L}_\varrho f)(t) \quad (t \in \mathbb{R}). \end{aligned}$$

Since $C_c^\infty(\mathbb{R}; H)$ is a core for $\mathbf{i}m + \varrho$ according to the second step, so is $\mathcal{S}(\mathbb{R}; H)$ and hence, $\mathcal{S}_\varrho(\mathbb{R}; H)$ is a core for $\mathcal{L}_\varrho^*(\mathbf{i}m + \varrho) \mathcal{L}_\varrho$, since \mathcal{L}_ϱ is a bijection from $\mathcal{S}_\varrho(\mathbb{R}; H)$ to $\mathcal{S}(\mathbb{R}; H)$ by Proposition A.2. Since $\partial_{0,\varrho}$ and $\mathcal{L}_\varrho^*(\mathbf{i}m + \varrho) \mathcal{L}_\varrho$ coincide on $\mathcal{S}_\varrho(\mathbb{R}; H)$, which is a core for both operators, we derive the assertion. \square

As an immediate consequence, we obtain the normality of $\partial_{0,\varrho}$.

Proposition 1.1.5. *Let $\varrho \in \mathbb{R}$. Then $\partial_{0,\varrho}$ is a normal operator with $\partial_{0,\varrho}^* = -\partial_{0,\varrho} + 2\varrho$, $\operatorname{Re} \partial_{0,\varrho} = \varrho$ and $\operatorname{Im} \partial_{0,\varrho} = \mathbf{i}(\varrho - \partial_{0,\varrho})$. Moreover, $\sigma(\partial_{0,\varrho}) = C\sigma(\partial_{0,\varrho}) = \mathbb{C}_{\operatorname{Re}=\varrho}$.*

Proof. The assertion follows immediately from the respective results for the multiplication operator $\mathbf{i}m + \varrho$ on $L_2(\mathbb{R}; H)$ and Proposition 1.1.4. \square

The latter proposition especially yields that $\partial_{0,\varrho}$ is continuously invertible if and only if $\varrho \neq 0$. In fact, the formula for $\partial_{0,\varrho}^{-1}$ differs for $\varrho > 0$ or $\varrho < 0$ as the next proposition shows.

Proposition 1.1.6. *Let $\varrho \neq 0$. Then $\partial_{0,\varrho}$ is continuously invertible with $\|\partial_{0,\varrho}^{-1}\| = \frac{1}{|\varrho|}$ and for $f \in H_\varrho(\mathbb{R}; H)$, $t \in \mathbb{R}$ we have that*

$$(\partial_{0,\varrho}^{-1} f)(t) = \begin{cases} \int_{-\infty}^t f(s) \, ds, & \text{if } \varrho > 0, \\ -\int_t^\infty f(s) \, ds, & \text{if } \varrho < 0. \end{cases}$$

Proof. The bounded invertibility and the equality $\|\partial_{0,\varrho}^{-1}\| = \frac{1}{|\varrho|}$ follow from Proposition 1.1.4.

Let now $\varphi \in C_c^\infty(\mathbb{R}; H)$. If $\varrho > 0$ we get that $\mu := \left(t \mapsto \int_{-\infty}^t \varphi(s) \, ds\right) \in H_\varrho(\mathbb{R}; H)$. Indeed,

$$\begin{aligned} \int_{\mathbb{R}} |\mu(t)|_H^2 e^{-2\varrho t} \, dt &= \int_{\mathbb{R}} \left| \int_{-\infty}^t \varphi(s) e^{-\frac{\varrho}{2}s} e^{\frac{\varrho}{2}s} \, ds \right|_H^2 e^{-2\varrho t} \, dt \\ &\leq \frac{1}{\varrho} \int_{\mathbb{R}} \int_{-\infty}^t |\varphi(s)|_H^2 e^{-\varrho s} \, ds e^{-\varrho t} \, dt \\ &= \frac{1}{\varrho} \int_{\mathbb{R}} |\varphi(s)|_H^2 e^{-\varrho s} \int_s^\infty e^{-\varrho t} \, dt \, ds \\ &= \frac{1}{\varrho^2} \int_{\mathbb{R}} |\varphi(s)|_H^2 e^{-2\varrho s} \, ds \end{aligned}$$

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$$= \frac{1}{\varrho^2} |\varphi|_{\varrho}^2.$$

Moreover, for each $\psi \in C_c^\infty(\mathbb{R}; H)$ we have

$$\begin{aligned} \langle \partial_{0,\varrho} \psi | \mu \rangle_{\varrho} &= \int_{\mathbb{R}} \langle \psi'(t) | \mu(t) \rangle_H e^{-2\varrho t} dt \\ &= - \int_{\mathbb{R}} \langle \psi(t) | \varphi(t) - 2\varrho \mu(t) \rangle_H e^{-2\varrho t} dt \\ &= \langle \psi | -\varphi + 2\varrho \mu \rangle_{\varrho} \end{aligned}$$

and thus $\mu \in D(\partial_{0,\varrho}^*) = D(\partial_{0,\varrho})$ with $-\varphi + 2\varrho \mu = \partial_{0,\varrho}^* \mu = -\partial_{0,\varrho} \mu + 2\varrho \mu$ which gives $\partial_{0,\varrho} \mu = \varphi$, i.e. $\mu = \partial_{0,\varrho}^{-1} \varphi$. The statement for $\varrho < 0$ follows by arguing analogously and thus, we omit it. \square

Remark 1.1.7. As $\partial_{0,\varrho}$ is continuously invertible if $\varrho \neq 0$, we may also equip $H_{\varrho}^1(\mathbb{R}; H)$ with the inner product

$$\langle f | g \rangle := \langle \partial_{0,\varrho} f | \partial_{0,\varrho} g \rangle_{\varrho} \quad (f, g \in H_{\varrho}^1(\mathbb{R}; H)),$$

which yields an equivalent norm on $H_{\varrho}^1(\mathbb{R}; H)$.

We conclude this section with a version of the Sobolev embedding theorem.

Proposition 1.1.8 (Sobolev embedding). *Let $\varrho \in \mathbb{R}$. We define the space*

$$C_{\varrho}(\mathbb{R}; H) := \left\{ f : \mathbb{R} \rightarrow H ; f \text{ continuous, } \sup_{t \in \mathbb{R}} |f(t) e^{-\varrho t}|_H < \infty \right\}$$

equipped with the norm

$$|f|_{\infty, \varrho} := \sup_{t \in \mathbb{R}} |f(t) e^{-\varrho t}|_H.$$

Moreover, we define $C_{\varrho,0}(\mathbb{R}; H) := \{ f \in C_{\varrho}(\mathbb{R}; H) ; f(t) e^{-\varrho t} \rightarrow 0 \quad (t \rightarrow \pm\infty) \}$. Then we have that $H_{\varrho}^1(\mathbb{R}; H) \hookrightarrow C_{\varrho,0}(\mathbb{R}; H)$.

Proof. Let $\varphi \in C_c^\infty(\mathbb{R}; H)$. For $\varrho > 0$ we compute

$$\begin{aligned} |\varphi(t)|_H &= \left| \int_{-\infty}^t \varphi'(s) ds \right|_H \\ &\leq \left(\int_{-\infty}^t |\varphi'(s)|_H^2 e^{-2\varrho s} ds \right)^{\frac{1}{2}} \frac{1}{\sqrt{2\varrho}} e^{\varrho t} \\ &\leq |\varphi|_{\varrho,1} \frac{1}{\sqrt{2\varrho}} e^{\varrho t} \quad (t \in \mathbb{R}), \end{aligned}$$

which yields the continuity of the mapping $\text{id} : C_c^\infty(\mathbb{R}; H) \subseteq H_{\varrho}^1(\mathbb{R}; H) \rightarrow C_{\varrho,0}(\mathbb{R}; H)$. Noting that $C_{\varrho,0}(\mathbb{R}; H)$ is a Banach space, we obtain the asserted embedding by continuous extension.

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If $\varrho < 0$, we compute

$$\begin{aligned} |\varphi(t)|_H &= \left| \int_t^\infty \varphi'(s) \, ds \right|_H \\ &\leq \left(\int_t^\infty |\varphi'(s)|_H^2 e^{-2\varrho s} \, ds \right)^{\frac{1}{2}} \frac{1}{\sqrt{2|\varrho|}} e^{\varrho t} \\ &\leq |\varphi|_{\varrho,1} \frac{1}{\sqrt{2|\varrho|}} e^{\varrho t} \end{aligned}$$

for each $t \in \mathbb{R}$ and so the assertion follows as above. For $\varrho = 0$ we estimate

$$\begin{aligned} |\varphi(t)|_H &\leq \int_{t-1}^t |\varphi(t) - \varphi(s)|_H \, ds + \int_{t-1}^t |\varphi(s)|_H \, ds \\ &\leq \int_{t-1}^t \left| \int_s^t \varphi'(r) \, dr \right|_H \, ds + |\varphi|_{L_2} \\ &\leq \int_{t-1}^t \left(\int_s^t |\varphi'(r)|_H^2 \, dr \right)^{\frac{1}{2}} \sqrt{t-s} \, ds + |\varphi|_{L_2} \\ &\leq |\varphi'|_{L_2} + |\varphi|_{L_2} \\ &\leq \sqrt{2} |\varphi|_{0,1}, \end{aligned}$$

which yields the assertion. \square

Remark 1.1.9. One can show a slightly stronger result than in Proposition 1.1.8. In fact, $H_\varrho^1(\mathbb{R}; H)$ is continuously embedded into an (exponentially weighted) space of Hölder continuous functions (see e.g. [Eva10, p. 282] for $\varrho = 0$, [PM11, Lemma 3.1.59] for $\varrho > 0$).

1.2. Evolutionary problems

With the Hilbert space setting developed in the previous section at hand, we are now able to define so-called evolutionary problems. Evolutionary problems, as they were introduced in [Pic09], consist of two equations with two unknowns u and w belonging to $H_\varrho(\mathbb{R}; H)$ for some $\varrho \in \mathbb{R}$. First,

$$\partial_{0,\varrho} w + Au = f,$$

where $f \in H_\varrho(\mathbb{R}; H)$ is an arbitrary source term and $A : D(A) \subseteq H \rightarrow H$ is a linear operator acting on H , which is extended to $H_\varrho(\mathbb{R}; H)$ in the canonical way by $(Au)(t) := Au(t)$ for $t \in \mathbb{R}$ and $u \in \{g \in H_\varrho(\mathbb{R}; H) ; g(t) \in D(A) \text{ for a.e. } t \in \mathbb{R}, (t \mapsto Ag(t)) \in H_\varrho(\mathbb{R}; H)\}$. The first equation is completed by a second one, which links u and w :

$$w = \mathcal{M}u,$$

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where $\mathcal{M} : D(\mathcal{M}) \subseteq H_\varrho(\mathbb{R}; H) \rightarrow H_\varrho(\mathbb{R}; H)$ is a linear operator. Substituting the second equation into the first one, we end up with an equation of the form

$$(\partial_{0,\varrho}\mathcal{M} + A)u = f \quad (1.1)$$

and we refer to (1.1) as an *evolutionary equation*. We focus here on operators $\mathcal{M} = M(\partial_{0,\varrho})$, defined as operator-valued functions of the normal operator $\partial_{0,\varrho}$, which in particular implies that \mathcal{M} commutes with translations in time. Thus, we are dealing with autonomous equations. The precise definition of $\mathcal{M} = M(\partial_{0,\varrho})$ is as follows.

Definition. Let $M : D(M) \subseteq \mathbb{C} \rightarrow L(H)$ analytic, $D(M)$ open such that $\mathbb{C}_{\operatorname{Re} > \varrho_0} \subseteq D(M)$ for some $\varrho_0 \in \mathbb{R}$. Then we call M a *linear material law*. Moreover, we define the set

$$S_M := \{\varrho \in \mathbb{R}; it + \varrho \in D(M) \text{ for almost every } t \in \mathbb{R}\}$$

and for each $\varrho \in S_M$ we define the operator $M(\operatorname{im} + \varrho) : D(M(\operatorname{im} + \varrho)) \subseteq L_2(\mathbb{R}; H) \rightarrow L_2(\mathbb{R}; H)$ as follows

$$(M(\operatorname{im} + \varrho)f)(t) := M(it + \varrho)f(t) \quad (t \in \mathbb{R})$$

for each $f \in D(M(\operatorname{im} + \varrho)) := \{g \in L_2(\mathbb{R}; H); (t \mapsto M(it + \varrho)g(t)) \in L_2(\mathbb{R}; H)\}$. Furthermore, we define the operator $M(\partial_{0,\varrho}) : D(M(\partial_{0,\varrho})) \subseteq H_\varrho(\mathbb{R}; H) \rightarrow H_\varrho(\mathbb{R}; H)$ by

$$M(\partial_{0,\varrho}) := \mathcal{L}_\varrho^* M(\operatorname{im} + \varrho) \mathcal{L}_\varrho$$

with its natural domain.

We state some elementary properties of linear material laws.

Proposition 1.2.1. *Let $M : D(M) \subseteq \mathbb{C} \rightarrow L(H)$ be a linear material law. Then for each $\varrho \in S_M$ the operators $M(\operatorname{im} + \varrho)$ and $M(\partial_{0,\varrho})$ are densely defined and closed.*

Proof. Let $\varrho \in S_M$. By unitary equivalence, it suffices to prove the asserted properties for the operator $M(\operatorname{im} + \varrho)$. For showing the density of $D(M(\operatorname{im} + \varrho))$, we set

$$D := \{t \in \mathbb{R}; it + \varrho \in D(M)\},$$

which is an open subset of \mathbb{R} with $\lambda(\mathbb{R} \setminus D) = 0$. The latter gives that $L_2(D; H) = L_2(\mathbb{R}; H)$ (more precisely, the mapping $L_2(\mathbb{R}; H) \ni f \mapsto f|_D \in L_2(D; H)$ is unitary). Moreover, since $C_c^\infty(D; H)$ is dense in $L_2(D; H)$, we infer the density of $C_c^\infty(D; H)$ in $L_2(\mathbb{R}; H)$. Finally, using that M is bounded on compact subsets of D , we have that $C_c^\infty(D; H) \subseteq D(M(\operatorname{im} + \varrho))$, and so the density of $D(M(\operatorname{im} + \varrho))$ follows.

It is left to prove the closedness of $M(\operatorname{im} + \varrho)$. For doing so, let $(f_n)_{n \in \mathbb{N}}$ in $D(M(\operatorname{im} + \varrho))$ with $f_n \rightarrow f$ and $M(\operatorname{im} + \varrho)f_n \rightarrow g$ in $L_2(\mathbb{R}; H)$ for some $f, g \in L_2(\mathbb{R}; H)$. Passing to a subsequence, we may assume without loss of generality that $f_n(t) \rightarrow f(t)$ and $M(it + \varrho)f_n(t) \rightarrow g(t)$ for almost every $t \in \mathbb{R}$. Since $M(it + \varrho)$ is bounded, we infer $M(it + \varrho)f(t) = g(t)$ for almost every $t \in \mathbb{R}$, which shows the assertion. \square

Using the notion of linear material laws, we consider evolutionary equations of the form

$$(\partial_{0,\varrho}M(\partial_{0,\varrho}) + A)u = f.$$

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We first address the question of well-posedness of such an equation, which we define as follows:

Definition. Let $M : D(M) \subseteq \mathbb{C} \rightarrow L(H)$ be a linear material law. Moreover, let $A : D(A) \subseteq H \rightarrow H$ be a densely defined closed linear operator. For $\varrho \in S_M$ we call the problem of finding a solution $u \in H_\varrho(\mathbb{R}; H)$ of

$$(\partial_{0,\varrho} M(\partial_{0,\varrho}) + A)u = f$$

for right-hand sides $f \in H_\varrho(\mathbb{R}; H)$ an *evolutionary problem associated with M and A* . We call such a problem *well-posed*, if there exists some $\varrho_1 \in \mathbb{R}$ such that

- (a) for each $z \in \mathbb{C}_{\text{Re} > \varrho_1} \cap D(M)$ the operator $zM(z) + A$ is boundedly invertible, and
- (b) the mapping $\mathbb{C}_{\text{Re} > \varrho_1} \cap D(M) \ni z \mapsto (zM(z) + A)^{-1} \in L(H)$ possesses a bounded and analytic extension to $\mathbb{C}_{\text{Re} > \varrho_1}$.

Moreover, we define the *abscissa of boundedness* $s_0(M, A)$ of a well-posed problem by

$$s_0(M, A) := \inf \{ \varrho_1 \in \mathbb{R} ; \text{(a) and (b) are satisfied} \}.$$

We now show that the well-posedness definition above indeed yields the unique solvability of the evolutionary problem and the continuous dependence of the solution on the given right-hand side.

Proposition 1.2.2. *Assume we have a well-posed evolutionary problem associated with M and A . Then for each $\varrho \in S_M$ with $\varrho > s_0(M, A)$ the operator $\partial_{0,\varrho} M(\partial_{0,\varrho}) + A$ is closable and $\overline{\partial_{0,\varrho} M(\partial_{0,\varrho}) + A}$ is boundedly invertible.*

Proof. For $\varrho \in S_M$ we define the multiplication operator

$$\begin{aligned} S_\varrho : D(S_\varrho) \subseteq L_2(\mathbb{R}; H) &\rightarrow L_2(\mathbb{R}; H) \\ u &\mapsto (t \mapsto ((it + \varrho)M(it + \varrho) + A)u(t)) \end{aligned}$$

with maximal domain $D(S_\varrho)$ given by

$$\{ u \in L_2(\mathbb{R}; H) ; u(t) \in D(A) \text{ for a.e. } t \in \mathbb{R}, ((i \cdot + \varrho)M(i \cdot + \varrho) + A)u(\cdot) \in L_2(\mathbb{R}; H) \}.$$

Then S_ϱ is closed. Indeed, let $(u_n)_{n \in \mathbb{N}}$ in $D(S_\varrho)$ with $u_n \rightarrow u$ and $S_\varrho u_n \rightarrow v$ in $L_2(\mathbb{R}; H)$ for some $u, v \in L_2(\mathbb{R}; H)$. Passing to a subsequence (without re-labeling), we infer that $u_n(t) \rightarrow u(t)$ and $(S_\varrho u_n)(t) \rightarrow v(t)$ for almost every $t \in \mathbb{R}$. Since $M(it + \varrho)$ is bounded, we derive that

$$Au_n(t) = (S_\varrho u_n)(t) - (it + \varrho)M(it + \varrho)u_n(t) \rightarrow v(t) - (it + \varrho)M(it + \varrho)u(t),$$

and thus, $u(t) \in D(A)$ for almost every $t \in \mathbb{R}$, since A is closed. Moreover, $(it + \varrho)M(it + \varrho)u(t) + Au(t) = v(t)$ for almost every $t \in \mathbb{R}$, which shows $u \in D(S_\varrho)$ and $S_\varrho u = v$. Since clearly

$$(im + \varrho)M(im + \varrho) + A \subseteq S_\varrho,$$

we infer the closability of $(im + \varrho)M(im + \varrho) + A$ and hence, by unitary equivalence we get the closability of $\partial_{0,\varrho} M(\partial_{0,\varrho}) + A$. We show $\overline{(im + \varrho)M(im + \varrho) + A} = S_\varrho$. By what we have shown above, we need to verify that $S_\varrho \subseteq \overline{(im + \varrho)M(im + \varrho) + A}$. So let $u \in D(S_\varrho)$. Then we set

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$u_n(t) := \chi_{[-n,n]}(t)u(t)$ for $n \in \mathbb{N}, t \in \mathbb{R}$. By dominated convergence, $u_n \rightarrow u$ and $S_\varrho u_n \rightarrow S_\varrho u$ in $L_2(\mathbb{R}; H)$. Moreover,

$$\begin{aligned} \int_{\mathbb{R}} |(it + \varrho)M(it + \varrho)u_n(t)|^2 dt &\leq \sup_{s \in [-n,n]} \|M(is + \varrho)\| \int_{-n}^n (t^2 + \varrho^2) |u(t)|^2 dt \\ &\leq \sup_{s \in [-n,n]} \|M(is + \varrho)\| (n^2 + \varrho^2) \|u\|_{L_2}^2, \end{aligned}$$

which yields $u_n \in D((im + \varrho)M(im + \varrho))$ for each $n \in \mathbb{N}$. Since also $u_n \in D(S_\varrho)$ it follows that $u_n \in D((im + \varrho)M(im + \varrho) + A)$. Moreover, using that

$$((im + \varrho)M(im + \varrho) + A)u_n = S_\varrho u_n \rightarrow S_\varrho u,$$

we derive $u \in D(\overline{(im + \varrho)M(im + \varrho) + A})$, which yields the assertion.

Using the conditions (a) and (b) in the definition of well-posed evolutionary problems, we derive that $S_\varrho = \overline{(im + \varrho)M(im + \varrho) + A}$ and hence, by unitary equivalence, $\partial_{0,\varrho}M(\partial_{0,\varrho}) + A$ is boundedly invertible for $\varrho > s_0(M, A)$. \square

Besides the unique existence of a solution and its continuous dependence on the given data, we want to address the causality of the associated solution operator. Since evolutionary equations are intended to model some physical phenomenon, where the argument of the unknown u will be interpreted as time, causality will be one of its crucial properties. In our framework we define causality as follows.

Definition. Let $T : D(T) \subseteq H_\varrho(\mathbb{R}; H) \rightarrow H_\varrho(\mathbb{R}; H)$ be a mapping for some $\varrho \in \mathbb{R}$. We say T is (*forward*) *causal*, if for each $u, v \in D(T)$ with $u = v$ on $] -\infty, t]$ for some $t \in \mathbb{R}$ we have that $Tu = Tv$ on $] -\infty, t]$.

For an abstract notion of causality we refer to [Sae70]. Moreover, we note that there is another notion of causality introduced in [Wau15a], which coincides with the definition above in case of closed operators and which has the benefit that the closure of a causal operator stays causal (which is not the case for the definition above).

We begin by stating some equivalent formulations of causality for certain classes of mappings. In order to fix some notation, we make the following definitions.

Definition. Let $\varrho \in \mathbb{R}$.

- (a) For $t \in \mathbb{R}$ we define the *cut-off operator* $\chi_{\mathbb{R}_{\leq t}}(m) : H_\varrho(\mathbb{R}; H) \rightarrow H_\varrho(\mathbb{R}; H)$ by setting $(\chi_{\mathbb{R}_{\leq t}}(m)f)(s) := \chi_{\mathbb{R}_{\leq t}}(s)f(s)$ for $f \in H_\varrho(\mathbb{R}; H), s \in \mathbb{R}$.
- (b) For $h \in \mathbb{R}$ let $\tau_h : H_\varrho(\mathbb{R}; H) \rightarrow H_\varrho(\mathbb{R}; H)$ be the *translation operator* defined by $(\tau_h f)(s) := f(s + h)$ for $f \in H_\varrho(\mathbb{R}; H), s \in \mathbb{R}$.
- (c) A mapping $T : D(T) \subseteq H_\varrho(\mathbb{R}; H) \rightarrow H_\varrho(\mathbb{R}; H)$ is called *translation-invariant*, if $\tau_h T = T\tau_h$ for each $h \in \mathbb{R}$.

Lemma 1.2.3. Let $\varrho \in \mathbb{R}$.

- (a) A mapping $T : H_\varrho(\mathbb{R}; H) \rightarrow H_\varrho(\mathbb{R}; H)$ is causal, if and only if $\chi_{\mathbb{R}_{\leq t}}(m)T\chi_{\mathbb{R}_{\leq t}}(m) = \chi_{\mathbb{R}_{\leq t}}(m)T$ for each $t \in \mathbb{R}$.

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- (b) A linear mapping $T : D(T) \subseteq H_\varrho(\mathbb{R}; H) \rightarrow H_\varrho(\mathbb{R}; H)$ is causal, if and only if for each $u \in D(T)$ with $u = 0$ on $] -\infty, t]$ for some $t \in \mathbb{R}$ it follows that $Tu = 0$ on $] -\infty, t]$.
- (c) A translation-invariant mapping $T : D(T) \subseteq H_\varrho(\mathbb{R}; H) \rightarrow H_\varrho(\mathbb{R}; H)$ is causal, if and only if for each $u, v \in D(T)$ with $u = v$ on $] -\infty, 0]$ it follows that $Tu = Tv$ on $] -\infty, 0]$. If in addition $D(T) = H_\varrho(\mathbb{R}; H)$, the latter is equivalent to $\chi_{\mathbb{R}_{\leq 0}}(\mathbf{m})T\chi_{\mathbb{R}_{\leq 0}}(\mathbf{m}) = \chi_{\mathbb{R}_{\leq 0}}(\mathbf{m})T$.

Proof. (a) Let T be causal, $u \in H_\varrho(\mathbb{R}; H)$ and $t \in \mathbb{R}$. Then $u = \chi_{\mathbb{R}_{\leq t}}(\mathbf{m})u$ on $] -\infty, t]$ and thus, $Tu = T\chi_{\mathbb{R}_{\leq t}}(\mathbf{m})u$ on $] -\infty, t]$ which gives $\chi_{\mathbb{R}_{\leq t}}(\mathbf{m})Tu = \chi_{\mathbb{R}_{\leq t}}(\mathbf{m})T\chi_{\mathbb{R}_{\leq t}}(\mathbf{m})u$. Assume now that $\chi_{\mathbb{R}_{\leq s}}(\mathbf{m})T\chi_{\mathbb{R}_{\leq s}}(\mathbf{m}) = \chi_{\mathbb{R}_{\leq s}}(\mathbf{m})T$ for each $s \in \mathbb{R}$ and take $u, v \in H_\varrho(\mathbb{R}; H)$ with $u = v$ on $] -\infty, t]$ for some $t \in \mathbb{R}$. Then,

$$\begin{aligned}\chi_{\mathbb{R}_{\leq t}}(\mathbf{m})Tu &= \chi_{\mathbb{R}_{\leq t}}(\mathbf{m})T\chi_{\mathbb{R}_{\leq t}}(\mathbf{m})u \\ &= \chi_{\mathbb{R}_{\leq t}}(\mathbf{m})T\chi_{\mathbb{R}_{\leq t}}(\mathbf{m})v \\ &= \chi_{\mathbb{R}_{\leq t}}(\mathbf{m})Tv\end{aligned}$$

and thus $Tu = Tv$ on $] -\infty, t]$.

- (b) This is clear, since for $u, v \in D(T)$ we have $u - v \in D(T)$ and $u = v$ and $Tu = Tv$ on $] -\infty, t]$ is equivalent to $u - v = 0$ and $T(u - v) = 0$ on $] -\infty, t]$, respectively.
- (c) Since $\chi_{\mathbb{R}_{\leq t}}(\mathbf{m}) = \tau_{-t}\chi_{\mathbb{R}_{\leq 0}}(\mathbf{m})\tau_t$ for each $t \in \mathbb{R}$, the first assertion follows. In the case $D(T) = \bar{H}_\varrho(\mathbb{R}; H)$ the assertion follows from (a). \square

For later purposes, we need some further properties of bounded analytic functions on half planes.

Proposition 1.2.4. *Let $T : \mathbb{C}_{\operatorname{Re} > \varrho_0} \rightarrow L(H)$ be bounded and analytic for some $\varrho_0 \in \mathbb{R}$. Then there exists a unique operator $T_{\varrho_0} \in L(L_2(\mathbb{R}; H))$ such that for $\varphi \in C_c^\infty(\mathbb{R}; H)$ we have that*

$$T(\mathbf{i} \mathbf{m} + \varrho)\mathcal{L}_\varrho\varphi \rightarrow T_{\varrho_0}\mathcal{L}_{\varrho_0}\varphi \quad (\varrho \rightarrow \varrho_0)$$

in $L_2(\mathbb{R}; H)$. Moreover, $\|T_{\varrho_0}\| \leq \|T\|_\infty$.

Proof. The uniqueness of such an operator is clear, since it is determined on $\mathcal{L}_{\varrho_0}[C_c^\infty(\mathbb{R}; H)]$, which is dense in $L_2(\mathbb{R}; H)$. Let now $\varphi \in C_c^\infty(\mathbb{R}; H)$ with $\operatorname{spt} \varphi \subseteq \mathbb{R}_{\geq 0}$. Then we have $\hat{\varphi} \in \mathcal{H}^2(\mathbb{C}_{\operatorname{Re} > 0}; H)$ by Corollary A.8. Since T is bounded, we have that

$$(z \mapsto T(z)\hat{\varphi}(z)) \in \mathcal{H}^2(\mathbb{C}_{\operatorname{Re} > 0}; H)$$

and thus, there is $f \in L_2(\mathbb{R}_{\geq 0}; H)$ such that

$$\mathcal{L}_\varrho f = T(\mathbf{i} \mathbf{m} + \varrho)\mathcal{L}_\varrho\varphi \quad (\varrho > \varrho_0).$$

Since $\mathcal{L}_\varrho f \rightarrow \mathcal{L}_{\varrho_0} f$ in $L_2(\mathbb{R}_{\geq 0}; H)$ as $\varrho \rightarrow \varrho_0$ by dominated convergence, the right-hand side converges in $L_2(\mathbb{R}; H)$ as well. Let now $\varphi \in C_c^\infty(\mathbb{R}; H)$. Then we set $a := \inf \operatorname{spt} \varphi$ and thus, we have that $\tau_a \varphi \in C_c^\infty(\mathbb{R}_{\geq 0})$. Moreover, since $\mathcal{L}_\varrho \tau_a \varphi = e^{(\mathbf{i} \mathbf{m} + \varrho)a} \mathcal{L}_\varrho \varphi$, we infer that

$$T(\mathbf{i} \mathbf{m} + \varrho)\mathcal{L}_\varrho\varphi = e^{-(\mathbf{i} \mathbf{m} + \varrho)a} T(\mathbf{i} \mathbf{m} + \varrho)\mathcal{L}_\varrho \tau_a \varphi$$

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and thus,

$$T_{\varrho_0} \mathcal{L}_{\varrho_0} \varphi := \lim_{\varrho \rightarrow \varrho_0} T(\mathrm{i} m + \varrho) \mathcal{L}_{\varrho} \varphi$$

is well-defined. Then, T_{ϱ_0} is obviously linear and

$$\begin{aligned} |T_{\varrho_0} \mathcal{L}_{\varrho_0} \varphi|_{L_2(\mathbb{R}; H)} &\leq \|T\|_{\infty} \liminf_{\varrho \rightarrow \varrho_0} |\mathcal{L}_{\varrho} \varphi|_{L_2(\mathbb{R}; H)} \\ &= \|T\|_{\infty} |\mathcal{L}_{\varrho_0} \varphi|_{L_2(\mathbb{R}; H)} \end{aligned}$$

and hence, it extends to a bounded operator $T_{\varrho_0} \in L(L_2(\mathbb{R}; H))$ with $\|T_{\varrho_0}\| \leq \|T\|_{\infty}$. \square

Definition. Let $T : \mathbb{C}_{\mathrm{Re} > \varrho_0} \rightarrow L(H)$ be bounded and analytic for some $\varrho_0 \in \mathbb{R}$. Then we define the operator

$$T(\partial_{0, \varrho_0}) := \mathcal{L}_{\varrho_0}^* T_{\varrho_0} \mathcal{L}_{\varrho_0}.$$

Remark 1.2.5. We note that $T_{\varrho_0} = T(\mathrm{i} m + \varrho_0)$ for a bounded continuous function $T : \mathbb{C}_{\mathrm{Re} \geq \varrho_0} \rightarrow L(H)$, which is analytic in $\mathbb{C}_{\mathrm{Re} > \varrho_0}$. Indeed, since

$$T(\mathrm{i} m + \varrho) f \rightarrow T(\mathrm{i} m + \varrho_0) f \quad (\varrho \rightarrow \varrho_0)$$

for each $f \in L_2(\mathbb{R}; H)$ by dominated convergence, it follows that

$$T(\mathrm{i} m + \varrho) \mathcal{L}_{\varrho} \varphi \rightarrow T(\mathrm{i} m + \varrho_0) \mathcal{L}_{\varrho_0} \varphi \quad (\varrho \rightarrow \varrho_0)$$

for each $\varphi \in C_c^{\infty}(\mathbb{R}; H)$, which yields $T_{\varrho_0} = T(\mathrm{i} m + \varrho_0)$.

We note that the operator T_{ϱ_0} in Proposition 1.2.4 does not need to be a multiplication operator. However, if we assume that H is separable, we get the following statement.

Lemma 1.2.6. *Let $T : \mathbb{C}_{\mathrm{Re} > \varrho_0} \rightarrow L(H)$ bounded and analytic for some $\varrho_0 \in \mathbb{R}$ and assume that H is separable. Then there exists $h : \mathbb{R} \rightarrow L(H)$ strongly measurable and bounded such that*

$$T_{\varrho_0} = h(m).$$

Proof. Choose $D \subseteq H$ dense and countable subspace. Let $y \in D$ and consider the function $f_y \in \bigcap_{\varrho \geq \varrho_0} H_{\varrho}(\mathbb{R}; H)$ given by

$$f_y(t) := \sqrt{2\pi} \chi_{\mathbb{R}_{\geq 0}}(t) e^{-(1-\varrho_0)t} y \quad (t \in \mathbb{R}).$$

Then $\widehat{f_y}(z) = \frac{1}{z+1-\varrho_0} y$ for $z \in \mathbb{C}_{\mathrm{Re} \geq \varrho_0}$. Then

$$T(\mathrm{i} t + \varrho) y = (\mathrm{i} t + \varrho + 1 - \varrho_0) T(\mathrm{i} t + \varrho) (\mathcal{L}_{\varrho} f_y)(t) \quad (t \in \mathbb{R})$$

and since $T(\mathrm{i} m + \varrho) \mathcal{L}_{\varrho} f_y \rightarrow T_{\varrho_0} \mathcal{L}_{\varrho_0} f_y$ in $L_2(\mathbb{R}; H)$ as $\varrho \rightarrow \varrho_0$, we find a sequence $(\varrho_n)_{n \in \mathbb{N}}$ with $\varrho_n \rightarrow \varrho_0$ a Lebesgue null-set $M_y \subseteq \mathbb{R}$ such that

$$T(\mathrm{i} t + \varrho_n) y \rightarrow (\mathrm{i} t + 1) (T_{\varrho_0} \mathcal{L}_{\varrho_0} f_y)(t) \quad (n \rightarrow \infty)$$

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for $t \in \mathbb{R} \setminus M_y$. We set $M := \bigcup_{y \in D} M_y$ and define

$$h(t)y := \begin{cases} (it + 1)(T_{\varrho_0} \mathcal{L}_{\varrho_0} f_y)(t) & \text{if } t \in \mathbb{R} \setminus M, \\ 0 & \text{otherwise,} \end{cases}$$

for $y \in D$. Then $h(t) : D \rightarrow H$ is linear and

$$|h(t)y|_H \leq \|T\|_\infty |y|_H \quad (y \in D)$$

for each $t \in \mathbb{R}$ and hence, it has a unique extension $h(t) \in L(H)$. Since $h(\cdot)y$ is measurable for each $y \in D$ we infer that $h : \mathbb{R} \rightarrow L(H)$ is strongly measurable and bounded. It is left to show $T_\varrho = h(m)$. For doing so, we take $\varphi \in C_c^\infty(\mathbb{R}; H)$. Moreover, we choose a subsequence (without re-labeling) $(\varrho_n)_{n \in \mathbb{N}}$ with $\varrho_n \rightarrow \varrho_0$ such that

$$T(it + \varrho_n)(\mathcal{L}_{\varrho_n} \varphi)(t) \rightarrow (T_{\varrho_0} \mathcal{L}_{\varrho_0} \varphi)(t)$$

and $(\mathcal{L}_{\varrho_n} \varphi)(t) \rightarrow (\mathcal{L}_{\varrho_0} \varphi)(t)$ for every $t \in \mathbb{R} \setminus N$, where N is a Lebesgue null-set. Let now $\varepsilon > 0$, $t \in \mathbb{R} \setminus (M \cup N)$ and choose $y \in D$ such that $|\mathcal{L}_{\varrho_0} \varphi(t) - y|_H < \varepsilon$. We choose $n_0 \in \mathbb{N}$ such that

$$\begin{aligned} |T(it + \varrho_n)y - h(t)y|_H &< \varepsilon, \\ |(\mathcal{L}_{\varrho_n} \varphi)(t) - (\mathcal{L}_{\varrho_0} \varphi)(t)|_H &< \varepsilon \end{aligned}$$

for each $n \geq n_0$. Then we have for $n \geq n_0$

$$\begin{aligned} &|T(it + \varrho_n)(\mathcal{L}_{\varrho_n} \varphi)(t) - h(t)(\mathcal{L}_{\varrho_0} \varphi)(t)|_H \\ &\leq |T(it + \varrho_n)((\mathcal{L}_{\varrho_n} \varphi)(t) - (\mathcal{L}_{\varrho_0} \varphi)(t))|_H + |T(it + \varrho_n)((\mathcal{L}_{\varrho_0} \varphi)(t) - y)|_H + \\ &\quad + |(T(it + \varrho_n) - h(t))y|_H + |h(t)(y - (\mathcal{L}_{\varrho_0} \varphi)(t))|_H \\ &\leq (3\|T\|_\infty + 1)\varepsilon, \end{aligned}$$

which yields

$$(T_{\varrho_0} \mathcal{L}_{\varrho_0} \varphi)(t) = h(t)(\mathcal{L}_{\varrho_0} \varphi)(t)$$

for $t \in \mathbb{R} \setminus (N \cup M)$ and thus, $T_{\varrho_0} = h(m)$. \square

Remark 1.2.7. We note that $h \notin L_\infty(\mathbb{R}; L(H))$ in general. The existence of a boundary function in $L_p(\mathbb{R}; X)$ for functions $\mathcal{H}^p(\mathbb{C}_{\text{Re}>0}; X)$ for some Banach space X and $1 \leq p \leq \infty$ was studied in [Buk81]. Banach spaces allowing such boundary functions are said to have the *analytic Radon-Nikodym property*. It was shown that if $c_0 \subseteq X$, then X fails to have this property and thus, we cannot expect a better measurability than asserted in Lemma 1.2.6. We note that if X is a Banach lattice, $c_0 \not\subseteq X$ is equivalent to the analytic Radon-Nikodym property [BD82, Theorem 1]. This in particular implies that the analytic Radon-Nikodym property is weaker than the Radon-Nikodym property, since $c_0 \not\subseteq L_1(0, 1)$ but $L_1(0, 1)$ does not satisfy the Radon-Nikodym property (see [DU77, p. 61]).

We need a further auxiliary result for bounded analytic functions.

Lemma 1.2.8. *Let $\mu, \nu \in \mathbb{R}$ with $\mu < \nu$ and set $U := \{z \in \mathbb{C}; \mu < \text{Re } z < \nu\}$. Moreover, let $T : \overline{U} \subseteq \mathbb{C} \rightarrow L(H)$ be continuous, bounded and analytic in U . Then for $f \in H_\mu(\mathbb{R}; H) \cap$*

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$H_\nu(\mathbb{R}; H)$ we have that

$$\mathcal{L}_\mu^* T(\mathbf{i} m + \mu) \mathcal{L}_\mu f = \mathcal{L}_\nu^* T(\mathbf{i} m + \nu) \mathcal{L}_\nu f.$$

Proof. By Lemma 1.1.2 there exists a sequence in $C_c^\infty(\mathbb{R}; H)$ converging to f in both spaces $H_\mu(\mathbb{R}; H)$ and $H_\nu(\mathbb{R}; H)$. Thus, due to the boundedness of T it suffices to prove the assertion for all $\varphi \in C_c^\infty(\mathbb{R}; H)$. So let $\varphi \in C_c^\infty(\mathbb{R}; H)$. For $t \in \mathbb{R}$ we define the function

$$f(z) := e^{zt} T(z) \widehat{\varphi}(z) \quad (z \in \overline{U}).$$

Then f is analytic in U and so,

$$\mathbf{i} \int_{-R}^R f(\mathbf{i} s + \mu) \, ds + \int_{\mu}^{\nu} f(\mathbf{i} R + \kappa) \, d\kappa - \mathbf{i} \int_{-R}^R f(\mathbf{i} s + \nu) \, ds - \int_{\mu}^{\nu} f(-\mathbf{i} R + \kappa) \, d\kappa = 0 \quad (1.2)$$

for each $R > 0$. Moreover, f is bounded on \overline{U} since

$$\begin{aligned} |f(z)|_H &\leq \|T\|_\infty \max\{e^{\mu t}, e^{\nu t}\} |\widehat{\varphi}(z)|_H \\ &\leq \|T\|_\infty \max\{e^{\mu t}, e^{\nu t}\} C |\varphi|_{L_1}, \end{aligned} \quad (1.3)$$

for each $z \in \overline{U}$, where $C := \frac{1}{\sqrt{2\pi}} \sup\{e^{-\kappa s}; \kappa \in [\mu, \nu], s \in \text{spt } \varphi\}$. Moreover, since $\widehat{\varphi}(\pm \mathbf{i} R + \kappa) \rightarrow 0$ as $R \rightarrow \infty$ for each $\kappa \in]\mu, \nu[$ by the Riemann-Lebesgue Lemma (see Remark A.1), we get $f(\pm \mathbf{i} R + \kappa) \rightarrow 0$ as $R \rightarrow \infty$ for each $\kappa \in]\varrho, \mu[$ by the first line of (1.3). Thus, the second and fourth term in (1.2) tend to 0 as $R \rightarrow \infty$ by dominated convergence. Moreover

$$\int_{-R}^R f(\mathbf{i} s + \mu) \, ds = \int_{-R}^R e^{(\mathbf{i} s + \mu)t} T(\mathbf{i} s + \mu) (\mathcal{L}_\mu \varphi)(s) \, ds = \sqrt{2\pi} (\mathcal{L}_\mu^* \chi_{[-R, R]}(m) T(\mathbf{i} m + \mu) \mathcal{L}_\mu \varphi)(t)$$

and the same for μ replaced by ν . Passing now to a suitable subsequence, (1.2) and the last equality yield

$$(\mathcal{L}_\mu^* T(\mathbf{i} m + \mu) \mathcal{L}_\mu \varphi)(t) = (\mathcal{L}_\nu^* T(\mathbf{i} m + \nu) \mathcal{L}_\nu \varphi)(t) \quad (t \in \mathbb{R} \text{ a.e.}),$$

which gives the assertion. \square

Theorem 1.2.9. *Let $\varrho_0 \in \mathbb{R}$ and $T : \mathbb{C}_{\text{Re} > \varrho_0} \rightarrow L(H)$ be analytic and bounded. Then for each $\varrho \geq \varrho_0$ the operator $T(\partial_{0, \varrho})$ is translation-invariant, causal and bounded with $\|T(\partial_{0, \varrho})\| \leq \|T\|_\infty$. Moreover, the operator is independent on the choice of ϱ in the sense that $T(\partial_{0, \varrho})f = T(\partial_{0, \nu})f$ for each $f \in H_\varrho(\mathbb{R}; H) \cap H_\nu(\mathbb{R}; H)$ and $\varrho, \nu \geq \varrho_0$.*

Proof. First, let $\varrho > \varrho_0$. Since $\mathcal{L}_\varrho \tau_h = e^{(\mathbf{i} m + \varrho)h} \mathcal{L}_\varrho$, we infer the translation-invariance of $T(\partial_{0, \varrho})$. For proving the causality, it suffices to prove that for $u \in H_\varrho(\mathbb{R}; H)$ with $u = 0$ on $] - \infty, 0]$ it follows that $T(\partial_{0, \varrho})u = 0$ on $] - \infty, 0]$ by Lemma 1.2.3. So let $u \in H_\varrho(\mathbb{R}; H)$ with $u = 0$ on $] - \infty, 0]$. By Corollary A.8 we get that $\widehat{u} \in \mathcal{H}^2(\mathbb{C}_{\text{Re} > \varrho}; H)$. As T is analytic and bounded, we get that $(z \mapsto T(z)\widehat{u}(z)) \in \mathcal{H}^2(\mathbb{C}_{\text{Re} > \varrho}; H)$, which yields $T(\partial_{0, \varrho})u = 0$ on $] - \infty, 0]$ again by Corollary A.8. The boundedness of $T(\partial_{0, \varrho})$ and the norm estimate are obvious.

Let $f \in H_\varrho(\mathbb{R}; H) \cap H_\nu(\mathbb{R}; H)$ and $\varrho < \nu$. If $\varrho > \varrho_0$ we infer $T(\partial_{0, \varrho})f = T(\partial_{0, \nu})f$ from Lemma

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1.2.8. Assume now $\varrho = \varrho_0$. We first prove that $T(\partial_{0,\varrho_0})\varphi = T(\partial_{0,\nu})\varphi$ for each $\nu > \varrho_0$ and $\varphi \in C_c^\infty(\mathbb{R}; H)$. We choose a sequence $(\varrho_n)_{n \in \mathbb{N}}$ in $] \varrho_0, \nu[$ such that $\varrho_n \rightarrow \varrho_0$. Then we have

$$(T(\partial_{0,\nu})\varphi)(t) = (\mathcal{L}_{\varrho_n}^* T(\text{i m} + \varrho_n) \mathcal{L}_{\varrho_n} \varphi)(t)$$

for every $n \in \mathbb{N}$ and almost every $t \in \mathbb{R}$ by what we have shown above. Since,

$$\begin{aligned} (\mathcal{L}_{\varrho_n}^* T(\text{i m} + \varrho_n) \mathcal{L}_{\varrho_n} \varphi)(t) &= e^{\varrho_n t} (\mathcal{F}^* T(\text{i m} + \varrho_n) \mathcal{L}_{\varrho_n} \varphi)(t) \\ &\rightarrow e^{\varrho_0 t} (\mathcal{F}^* T_{\varrho_0} \mathcal{L}_{\varrho_0} \varphi)(t) \\ &= (T(\partial_{0,\varrho_0})\varphi)(t) \end{aligned}$$

for almost every $t \in \mathbb{R}$, we derive the assertion. Again, by Lemma 1.1.2 we obtain $T(\partial_{0,\varrho_0})f = T(\partial_{0,\nu})f$ for each $f \in H_{\varrho_0}(\mathbb{R}; H) \cap H_\nu(\mathbb{R}; H)$. In particular, we get for each $\varphi \in C_c^\infty(\mathbb{R}; H)$ and $h \in \mathbb{R}$

$$T(\partial_{0,\varrho_0})\tau_h \varphi = T(\partial_{0,\nu})\tau_h \varphi = \tau_h T(\partial_{0,\nu})\varphi = \tau_h T(\partial_{0,\varrho_0})\varphi,$$

and thus, the translation-invariance of $T(\partial_{0,\varrho_0})$ follows by continuous extension. Moreover, for $\varphi \in C_c^\infty(\mathbb{R}_{>0}; H)$ we have that

$$\text{spt } T(\partial_{0,\varrho_0})\varphi = \text{spt } T(\partial_{0,\nu})\varphi \subseteq \mathbb{R}_{\geq 0},$$

and hence, causality follows again by continuous extension. \square

The latter theorem shows that analytic and bounded mappings on a right half plane induce a family of bounded, causal and translation-invariant operators on $H_\varrho(\mathbb{R}; H)$, which are independent of the particular choice of ϱ . In particular, this result applies to the solution operator of a well-posed evolutionary problem.

Corollary 1.2.10. *Consider a well-posed evolutionary problem associated with M and A . Then for each $\varrho \in S_M$ with $\varrho > s_0(M, A)$, the operator $\left(\overline{\partial_{0,\varrho} M(\partial_{0,\varrho}) + A}\right)^{-1}$ is causal and independent of the parameter ϱ in the sense that*

$$\left(\overline{\partial_{0,\varrho} M(\partial_{0,\varrho}) + A}\right)^{-1} f = \left(\overline{\partial_{0,\nu} M(\partial_{0,\nu}) + A}\right)^{-1} f$$

for each $f \in H_\varrho(\mathbb{R}; H) \cap H_\nu(\mathbb{R}; H)$ and $\varrho, \nu \in S_M$ with $\varrho, \nu > s_0(M, A)$.

Proof. By assumption there exists a bounded analytic mapping $T : \mathbb{C}_{\text{Re} > \varrho_0} \rightarrow L(H)$ for each $\varrho_0 > s_0(M, A)$ such that $T(z) = (zM(z) + A)^{-1}$ for each $z \in D(M)$. In particular, for $\varrho \in S_M$ we derive

$$T(\partial_{0,\varrho}) = \left(\overline{\partial_{0,\varrho} M(\partial_{0,\varrho}) + A}\right)^{-1}$$

and hence, the assertion follows from Theorem 1.2.9. \square

Summarizing, we have shown that our definition of well-posedness yields a bounded solution operator, which is causal and independent of the particular choice of the parameter ϱ in the sense of Theorem 1.2.9. The latter two properties strongly rely on the assumption (b) in the definition of well-posedness. We now show that assumption (b) is not only sufficient for

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causality and independence of ϱ , but also necessary, that is, we prove a converse statement of Theorem 1.2.9. For doing so, we need the following representation result.

Theorem 1.2.11. *Let $T : L_2(\mathbb{R}; H) \rightarrow L_2(\mathbb{R}; H)$ a bounded, translation-invariant, causal linear operator. Then there exists an analytic and bounded mapping $N : \mathbb{C}_{\text{Re}>0} \rightarrow L(H)$ satisfying $\|N\|_\infty \leq \|T\|$, such that for each $f \in L_2(\mathbb{R}_{\geq 0}; H)$ one has*

$$\widehat{Tf}(z) = N(z)\widehat{f}(z) \quad (z \in \mathbb{C}_{\text{Re}>0}).$$

The latter result is a special case of [FS55, Theorem 2], where also the case $L_2(\mathbb{R}^n; H)$ for $n > 1$ is considered and causality is defined with respect to a closed convex cone in \mathbb{R}^n . However, for our purposes it suffices to consider the case $n = 1$. For proving the latter theorem we follow the rationale given in [Wei91]. We begin with the following lemma.

Lemma 1.2.12. *Let $f \in L_2(\mathbb{R}_{\geq 0}; H)$. We assume there exists $z \in \mathbb{C}_{\text{Re}>0}$ such that for each $h \geq 0$ we have that*

$$f(t+h) = e^{-zh} f(t) \quad (t \in \mathbb{R}_{\geq 0} \text{ a.e.}).$$

Then $f \in L_1(\mathbb{R}_{\geq 0}; H)$ and

$$f(t) = e^{-zt} z \int_0^\infty f(s) \, ds \quad (t \in \mathbb{R}_{\geq 0} \text{ a.e.}).$$

Proof. We first show that $f \in L_1(\mathbb{R}_{\geq 0}; H)$. For doing so, we compute

$$\begin{aligned} \int_0^\infty |f(t)|_H \, dt &= \sum_{n=0}^\infty \int_n^{n+1} |f(t)|_H \, dt \\ &= \sum_{n=0}^\infty \int_0^1 |f(t+n)|_H \, dt \\ &= \sum_{n=0}^\infty \int_0^1 e^{-(\text{Re } z)n} |f(t)|_H \, dt \\ &\leq \|f\|_{L_2} \frac{1}{1 - e^{-\text{Re } z}} < \infty. \end{aligned}$$

We define the function $F(t) := \int_t^\infty f(r) \, dr$ for $t \in \mathbb{R}_{\geq 0}$, which is well-defined and continuous, since $f \in L_1(\mathbb{R}_{\geq 0}; H)$. Then we obtain for each $t \geq 0$

$$\begin{aligned} F(t) &= \int_t^\infty f(s) \, ds \\ &= \int_0^\infty f(s+t) \, ds \\ &= e^{-zt} F(0) \end{aligned}$$

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for each $t \in \mathbb{R}_{\geq 0}$, which proves that F is continuously differentiable with $F'(t) = e^{-zt}(-zF(0))$. On the other hand, Theorem B.5 gives

$$F'(t) = -f(t) \quad (t \in \mathbb{R}_{\geq 0} \text{ a.e.})$$

and hence, the assertion follows. \square

With this preparation we are able to prove Theorem 1.2.11.

Proof of Theorem 1.2.11. Let $z \in \mathbb{C}_{\operatorname{Re}>0}$, $y \in H$ and define

$$g_{z,y}(t) := \chi_{\mathbb{R}_{\geq 0}}(t) e^{-zt} y \quad (t \in \mathbb{R}).$$

Then for $h \geq 0$ we have $(\chi_{\mathbb{R}_{\geq 0}}(\mathfrak{m})\tau_h g_{z,y})(t) = \chi_{\mathbb{R}_{\geq 0}}(t)g_{z,y}(t+h) = \chi_{\mathbb{R}_{\geq 0}}(t) e^{-z(t+h)} y = e^{-zh} g_{z,y}(t)$ for $t \in \mathbb{R}$. Since T is causal we have $\chi_{\mathbb{R}_{\leq 0}}(\mathfrak{m})T = \chi_{\mathbb{R}_{\leq 0}}(\mathfrak{m})T\chi_{\mathbb{R}_{\leq 0}}(\mathfrak{m})$ by Lemma 1.2.3 and hence,

$$\chi_{\mathbb{R}_{\geq 0}}(\mathfrak{m})T\chi_{\mathbb{R}_{\geq 0}}(\mathfrak{m}) = T\chi_{\mathbb{R}_{\geq 0}}(\mathfrak{m}) - \chi_{\mathbb{R}_{\leq 0}}(\mathfrak{m})T\chi_{\mathbb{R}_{\geq 0}}(\mathfrak{m}) = T\chi_{\mathbb{R}_{\geq 0}}(\mathfrak{m}).$$

The latter gives $\chi_{\mathbb{R}_{\geq 0}}(\mathfrak{m})T^*\chi_{\mathbb{R}_{\geq 0}}(\mathfrak{m}) = \chi_{\mathbb{R}_{\geq 0}}(\mathfrak{m})T^*$ and hence we obtain, by using that T^* is translation-invariant as well,

$$\begin{aligned} \chi_{\mathbb{R}_{\geq 0}}(\mathfrak{m})\tau_h\chi_{\mathbb{R}_{\geq 0}}(\mathfrak{m})T^*g_{z,y} &= \chi_{\mathbb{R}_{\geq 0}}(\mathfrak{m})\chi_{\mathbb{R}_{\geq -h}}(\mathfrak{m})\tau_hT^*g_{z,y} \\ &= \chi_{\mathbb{R}_{\geq 0}}(\mathfrak{m})T^*\tau_hg_{z,y} \\ &= \chi_{\mathbb{R}_{\geq 0}}(\mathfrak{m})T^*\chi_{\mathbb{R}_{\geq 0}}(\mathfrak{m})\tau_hg_{z,y} \\ &= \chi_{\mathbb{R}_{\geq 0}}(\mathfrak{m})T^*e^{-zh}g_{z,y} \\ &= e^{-zh}\chi_{\mathbb{R}_{\geq 0}}(\mathfrak{m})T^*g_{z,y}, \end{aligned}$$

for each $h \geq 0$. Thus, the function $\chi_{\mathbb{R}_{\geq 0}}(\mathfrak{m})T^*g_{z,y}$ satisfies the hypothesis of Lemma 1.2.12 and hence,

$$(T^*g_{z,y})(t) = e^{-zt} z \int_0^\infty (T^*g_{z,y})(s) \, ds \quad (t \in \mathbb{R}_{\geq 0} \text{ a.e.}). \quad (1.4)$$

For $z \in \mathbb{C}_{\operatorname{Re}>0}$ we consider now the mapping

$$\begin{aligned} L(z) : H &\rightarrow H \\ y &\mapsto z \int_0^\infty (T^*g_{z,y})(s) \, ds. \end{aligned}$$

Then, $L(z)$ is linear, since for $w, y \in H$ and $\lambda \in \mathbb{C}$ we have that $g_{z,\lambda w+y} = \lambda g_{z,w} + g_{z,y}$ and hence, the linearity of T^* implies the linearity of $L(z)$. Furthermore $L(z)$ is bounded with

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$\|L(z)\| \leq \|T\|$. Indeed, for $y \in H$ Equation (1.4) gives

$$\begin{aligned} |T^* g_{z,y}|_{L_2(\mathbb{R}_{\geq 0}; H)}^2 &= \int_0^\infty |(T^* g_{z,y})(t)|_H^2 dt \\ &= \int_0^\infty |e^{-zt} L(z)y|_H^2 dt \\ &= |L(z)y|_H^2 \frac{1}{2 \operatorname{Re} z} \end{aligned}$$

and hence,

$$\begin{aligned} \frac{1}{\sqrt{2 \operatorname{Re} z}} |L(z)y|_H &= |T^* g_{z,y}|_{L_2(\mathbb{R}_{\geq 0}; H)} \\ &\leq \|T^*\| |g_{z,y}|_{L_2} \\ &= \|T\| \frac{1}{\sqrt{2 \operatorname{Re} z}} |y|_H. \end{aligned}$$

Summarizing, we have shown that the mapping $L : \mathbb{C}_{\operatorname{Re} > 0} \rightarrow L(H)$ is well-defined and bounded with $\|L\|_\infty \leq \|T\|$. Let now $f \in L_2(\mathbb{R}_{\geq 0}; H)$. Then for $z \in \mathbb{C}_{\operatorname{Re} > 0}$ and $y \in H$ we have (note that $Tf = 0$ on $\mathbb{R}_{\leq 0}$ due to causality)

$$\begin{aligned} \langle y | \widehat{T}f(z) \rangle_H &= \left\langle y \left| \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}_{\geq 0}} e^{-zs} (Tf)(s) ds \right. \right\rangle_H \\ &= \frac{1}{\sqrt{2\pi}} \langle g_{z^*, y} | Tf \rangle_{L_2(\mathbb{R}_{\geq 0}; H)} \\ &= \frac{1}{\sqrt{2\pi}} \langle T^* g_{z^*, y} | f \rangle_{L_2(\mathbb{R}_{\geq 0}; H)} \\ &= \frac{1}{\sqrt{2\pi}} \langle L(z^*) y e^{-z^* m} | f \rangle_{L_2(\mathbb{R}_{\geq 0}; H)} \\ &= \langle L(z^*) y | \widehat{f}(z) \rangle_H \\ &= \langle y | L(z^*)^* \widehat{f}(z) \rangle_H, \end{aligned}$$

and thus

$$\widehat{T}f(z) = N(z) \widehat{f}(z) \quad (z \in \mathbb{C}_{\operatorname{Re} > 0})$$

with $N(z) := L(z^*)^*$. It remains to show that $z \mapsto N(z) \in L(H)$ is analytic. This however follows easily by applying the latter formula to $f = \sqrt{2\pi} g_{1,y}$ for $y \in H$. Then $\widehat{f}(z) = \frac{1}{z+1} y$ for $z \in \mathbb{C}_{\operatorname{Re} > 0}$ and thus,

$$N(z)y = (z+1) \widehat{T}f(z).$$

As the right-hand side is analytic on $\mathbb{C}_{\operatorname{Re} > 0}$, so is the left-hand side and hence, the analyticity of $z \mapsto N(z)$ follows from [HP57, Theorem 3.10.1]. \square

Indeed, Theorem 1.2.11 yields the converse of Theorem 1.2.9 in the following sense.

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Proposition 1.2.13. *Let $\varrho_0 \in \mathbb{R}$ and $\mathcal{T} : \bigcap_{\varrho \geq \varrho_0} H_\varrho(\mathbb{R}; H) \rightarrow \bigcap_{\varrho \geq \varrho_0} H_\varrho(\mathbb{R}; H)$ such that for each $\varrho \geq \varrho_0$, \mathcal{T} has a bounded, translation-invariant and causal extension $\mathcal{T}_\varrho \in L(H_\varrho(\mathbb{R}; H))$ and $\sup_{\varrho \geq \varrho_0} \|\mathcal{T}_\varrho\| < \infty$. Then there exists $T : \mathbb{C}_{\text{Re} > \varrho_0} \rightarrow L(H)$ analytic and bounded such that $\mathcal{T}_\varrho = T(\partial_{0, \varrho})$ for each $\varrho \geq \varrho_0$.*

Proof. Let $\varrho > \varrho_0$. Then, $S_\varrho := e^{-\varrho m} \mathcal{T}_\varrho (e^{-\varrho m})^{-1} : L_2(\mathbb{R}; H) \rightarrow L_2(\mathbb{R}; H)$ is bounded, causal and translation-invariant, where $e^{-\varrho m} : H_\varrho(\mathbb{R}; H) \rightarrow L_2(\mathbb{R}; H)$ is the unitary mapping defined by $(e^{-\varrho m} f)(t) := e^{-\varrho t} f(t)$ for each $t \in \mathbb{R}, f \in H_\varrho(\mathbb{R}; H)$. Thus, by Theorem 1.2.11 there is an analytic function $N_\varrho : \mathbb{C}_{\text{Re} > 0} \rightarrow L(H)$ with $\|N_\varrho\|_\infty \leq \|\mathcal{T}_\varrho\| = \|S_\varrho\|$ and

$$\widehat{S_\varrho f}(z) = N_\varrho(z) \widehat{f}(z) \quad (z \in \mathbb{C}_{\text{Re} > 0})$$

for each $f \in L_2(\mathbb{R}_{\geq 0}; H)$. Let $u \in \bigcap_{\varrho > \varrho_0} H_\varrho(\mathbb{R}; H)$ with $u = 0$ on $\mathbb{R}_{\leq 0}$. Then we obtain

$$(\widehat{e^{-\varrho m} \mathcal{T} u})(z) = (\widehat{S_\varrho e^{-\varrho m} u})(z) = N_\varrho(z) (\widehat{e^{-\varrho m} u})(z) \quad (z \in \mathbb{C}_{\text{Re} > 0}),$$

or in other words (set $z = it + \nu - \varrho$ for $\nu > \varrho$):

$$\mathcal{L}_\nu \mathcal{T} u = N_\varrho(it + \nu - \varrho) \mathcal{L}_\nu u, \quad (1.5)$$

for each $\nu > \varrho$. Let now $\varrho' > \varrho_0$ and set $u(t) := \sqrt{2\pi} \chi_{\mathbb{R}_{\geq 0}}(t) e^{\varrho_0 t} x$ for some $x \in H$ and $t \in \mathbb{R}$. Then the above gives for $\nu > \max\{\varrho, \varrho'\}$ and $t \in \mathbb{R}$

$$\begin{aligned} \frac{1}{it + \nu - \varrho_0} N_\varrho(it + \nu - \varrho) x &= N_\varrho(it + \nu - \varrho) (\mathcal{L}_\nu u)(t) \\ &= (\mathcal{L}_\nu \mathcal{T} u)(t) \\ &= N_{\varrho'}(it + \nu - \varrho') (\mathcal{L}_\nu u)(t) \\ &= \frac{1}{it + \nu - \varrho_0} N_{\varrho'}(it + \nu - \varrho') x, \end{aligned}$$

which yields

$$N_\varrho(z - \varrho) = N_{\varrho'}(z - \varrho') \quad (z \in \mathbb{C}_{\text{Re} > \max\{\varrho, \varrho'\}}). \quad (1.6)$$

We define the function

$$\begin{aligned} T : \mathbb{C}_{\text{Re} > \varrho_0} &\rightarrow L(H) \\ z &\mapsto N_{\frac{1}{2}(\text{Re } z + \varrho_0)} \left(z - \frac{1}{2}(\text{Re } z + \varrho_0) \right). \end{aligned}$$

Then T is bounded, since $|T(z)| \leq \|N_{\frac{1}{2}(\text{Re } z + \varrho_0)}\|_\infty \leq \|\mathcal{T}_{\frac{1}{2}(\text{Re } z + \varrho_0)}\| \leq \sup_{\varrho > \varrho_0} \|\mathcal{T}_\varrho\|$ for each $z \in \mathbb{C}_{\text{Re} > \varrho_0}$. Moreover, T is analytic. Indeed, for $z \in \mathbb{C}_{\text{Re} > \varrho_0}$ and $r := \frac{1}{2}(\text{Re } z - \varrho_0)$ we have that

$$T(z') = N_{\frac{1}{2}(\text{Re } z' + \varrho_0)} \left(z' - \frac{1}{2}(\text{Re } z' + \varrho_0) \right) = N_{\text{Re } z - r}(z' - (\text{Re } z - r)) \quad (z' \in B_{\mathbb{C}}(z, r))$$

by (1.6) and so the analyticity of T on $B_{\mathbb{C}}(z, r)$ follows from the analyticity of $N_{\text{Re } z - r}$ on $\mathbb{C}_{\text{Re} > 0}$. It remains to show $\mathcal{T}_\varrho = T(\partial_{0, \varrho})$ for each $\varrho > \varrho_0$. So let $\varrho > \varrho_0$ and $\varphi \in C_c^\infty(\mathbb{R}; H)$.

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We set $h := \inf \text{spt } \varphi$ and get $\text{spt } \tau_h \varphi \subseteq \mathbb{R}_{\geq 0}$. Equality (1.5) now gives

$$\begin{aligned} \mathcal{L}_\varrho \mathcal{T} \tau_h \varphi &= N_{\frac{1}{2}(\varrho + \varrho_0)} \left(i m + \varrho - \frac{1}{2}(\varrho + \varrho_0) \right) \mathcal{L}_\varrho \tau_h \varphi \\ &= T(i m + \varrho) \mathcal{L}_\varrho \tau_h \varphi. \end{aligned}$$

Using the translation-invariance of \mathcal{T} and the fact that $\mathcal{L}_\varrho \tau_h = e^{(i t + \varrho) h} \mathcal{L}_\varrho$ we get

$$\mathcal{L}_\varrho \mathcal{T} \varphi = T(i m + \varrho) \mathcal{L}_\varrho \varphi.$$

Since $C_c^\infty(\mathbb{R}; H)$ is dense in $H_\varrho(\mathbb{R}; H)$, we get $\mathcal{T}_\varrho = T(\partial_{0, \varrho})$ for each $\varrho > \varrho_0$. Moreover, since

$$T(\partial_{0, \varrho_0}) \varphi = T(\partial_{0, \varrho}) \varphi = \mathcal{T} \varphi,$$

we also get the assertion for $\varrho = \varrho_0$. □

We conclude this section by giving some examples for operators A and mappings M yielding a well-posed evolutionary problem. One important class of operators, which will be frequently used in the forthcoming sections is the class of m -accretive operators, which are defined as follows.

Definition. Let $A : D(A) \subseteq H \rightarrow H$ be linear. The operator A is called *accretive*, if for each $x \in D(A)$ we have

$$\text{Re} \langle Ax | x \rangle_H \geq 0.$$

Moreover, A is called *strictly accretive*, if there exists some $c > 0$ such that $A - c$ is accretive. Finally, A is called *m -accretive*, if A is accretive and there is some $\lambda \in \mathbb{C}_{\text{Re} > 0}$ such that $\lambda + A$ is onto.

Remark 1.2.14. The notion of accretive operators has a natural extension to operators on Banach spaces and also to non-linear operators, or even relations. For a deeper study of accretive and m -accretive operators we refer to the monographs [Bre71, Sho97, HP97]. The letter m in the notion m -accretive refers to the word maximal. The reason is, that A is m -accretive if and only if it is accretive and has no proper accretive extension in the set of binary relations on H . This equivalence is known as Minty's Theorem, [Min62].

We now show some useful properties of m -accretive operators.

Proposition 1.2.15. *Let $A : D(A) \subseteq H \rightarrow H$ be m -accretive. Then A is densely defined and closed. Moreover, $\mu + A$ is boundedly invertible for each $\mu \in \mathbb{C}_{\text{Re} > 0}$ and $\|(\mu + A)^{-1}\| \leq \frac{1}{\text{Re } \mu}$.*

Proof. Let $y \in D(A)^\perp$. By assumption, there is $\lambda \in \mathbb{C}_{\text{Re} > 0}$ and $x \in D(A)$ with $y = \lambda x + Ax$. Then we get

$$0 = \text{Re} \langle x | y \rangle_H = \text{Re} \langle x | \lambda x + Ax \rangle_H \geq \text{Re } \lambda |x|_H^2,$$

due to the accretivity of A . The latter gives $x = 0$, and consequently $y = 0$, which proves the density of $D(A)$. Let now $\mu \in \mathbb{C}_{\text{Re} > 0}$. Then we estimate

$$\text{Re} \langle x | \mu x + Ax \rangle_H \geq \text{Re } \mu |x|_H^2,$$

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and thus, $\mu + A$ is one-to-one and the inverse $(\mu + A)^{-1} : R(\mu + A) \subseteq H \rightarrow H$ is bounded with $\|(\mu + A)^{-1}\| \leq \frac{1}{\operatorname{Re} \mu}$. Moreover, by assumption, there is $\lambda \in \mathbb{C}_{\operatorname{Re} > 0}$ such that $\lambda + A$ is onto. Hence, $\lambda \in \varrho(A)$, which in particular implies the closedness of $(\lambda + A)^{-1}$ and hence of A . Furthermore, since $\lambda \in \varrho(A)$ we get

$$B(\lambda, \operatorname{Re} \lambda) \subseteq B(\lambda, \|(\lambda + A)^{-1}\|^{-1}) \subseteq \varrho(A),$$

which gives $\mathbb{C}_{\operatorname{Re} > 0} \subseteq \varrho(A)$ by induction. \square

Lemma 1.2.16. *Let $T \in L(H)$ such that*

$$\exists c > 0 \forall x \in H : \operatorname{Re} \langle Tx | x \rangle_H \geq c |x|_H^2.$$

Then T is boundedly invertible and

$$\operatorname{Re} \langle T^{-1}x | x \rangle_H \geq \frac{c}{\|T\|^2} |x|_H^2$$

for each $x \in H$.

Proof. The strict accretivity of T implies the injectivity of T . Moreover, the same holds for T^* and hence, T has dense range. Furthermore, $\|T^{-1}\|$ is bounded by $\frac{1}{c}$ and hence, the range of T is closed, proving the bounded invertibility of T . Let now $x \in H$. Then we estimate

$$|x|_H^2 = |TT^{-1}x|_H^2 \leq \|T\|^2 |T^{-1}x|_H^2 \leq \frac{\|T\|^2}{c} \operatorname{Re} \langle TT^{-1}x | T^{-1}x \rangle_H = \frac{\|T\|^2}{c} \operatorname{Re} \langle T^{-1}x | x \rangle_H. \quad \square$$

We also need a simple perturbation result for m -accretive operators.

Proposition 1.2.17. *Let $A : D(A) \subseteq H \rightarrow H$ m -accretive and $B \in L(H)$ accretive. Then $B + A$ is m -accretive.*

Proof. The accretivity of $B + A$ is obvious. Moreover, by Proposition 1.2.15 we have $\mathbb{C}_{\operatorname{Re} > 0} \subseteq \varrho(A)$. In particular, $\|B\| + 1 \in \varrho(A)$. Let $y \in H$. Then there is $x \in H$ with

$$(\|B\| + 1 + A)^{-1}(y - Bx) = x$$

by the contraction mapping theorem (note that $\|(\|B\| + 1 + A)^{-1}\| \leq \frac{1}{\|B\| + 1}$ by Proposition 1.2.15). Since the left-hand side belongs to $D(A)$, we infer $x \in D(A)$ with

$$(\|B\| + 1 + A)x = y - Bx$$

and thus, $(\|B\| + 1 + B + A)x = y$, which shows that $\|B\| + 1 + (B + A)$ is onto. Thus, $B + A$ is m -accretive. \square

With the help of the latter two proposition, we can provide a class of well-posed evolutionary problems.

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Proposition 1.2.18. *Let $M : D(M) \subseteq \mathbb{C} \rightarrow L(H)$ be a linear material law and $A : D(A) \subseteq H \rightarrow H$ be m -accretive. Moreover, assume there is $\varrho_0 \in \mathbb{R}$ and $c > 0$ such that*

$$\forall z \in \mathbb{C}_{\operatorname{Re} > \varrho_0} \cap D(M), x \in H : \operatorname{Re} \langle zM(z)x|x \rangle_H \geq c|x|_H^2. \quad (1.7)$$

Then the evolutionary problem associated with M and A is well-posed.

Proof. The estimate (1.7) states that $zM(z) - c$ is accretive for each $z \in \mathbb{C}_{\operatorname{Re} > \varrho_0} \cap D(M)$. Then, by Proposition 1.2.17 $zM(z) - c + A$ is m -accretive for each $z \in \mathbb{C}_{\operatorname{Re} > \varrho_0} \cap D(M)$. Hence, by Proposition 1.2.15, $zM(z) + A$ is boundedly invertible with $\|(zM(z) + A)^{-1}\| \leq \frac{1}{c}$ for each $z \in \mathbb{C}_{\operatorname{Re} > \varrho_0} \cap D(M)$. Choosing now $\varrho_1 \geq \varrho_0$ such that $\mathbb{C}_{\operatorname{Re} > \varrho_1} \subseteq D(M)$ we get that

$$\mathbb{C}_{\operatorname{Re} > \varrho_1} \ni z \mapsto (zM(z) + A)^{-1} \in L(H)$$

is an analytic and bounded function. Thus, both conditions for well-posedness of the evolutionary problem hold. \square

Remark 1.2.19. The latter proposition was first formulated in [Pic09, Solution Theory], where the mapping M was assumed to be bounded and A was skew-selfadjoint.

Finally, we consider a particular class of material laws M , which satisfy (1.7).

Proposition 1.2.20. *Let $M_0, M_1 \in L(H)$. Assume that M_0 is selfadjoint and that there exist $c_0, c_1 > 0$ with*

$$\begin{aligned} \langle M_0 x | x \rangle_H &\geq c_0 |x|_H^2 \quad (x \in R(M_0)), \\ \operatorname{Re} \langle M_1 x' | x' \rangle_H &\geq c_1 |x'|_H^2 \quad (x' \in N(M_0)). \end{aligned}$$

Then, $M : \mathbb{C} \setminus \{0\} \rightarrow L(H)$ defined by $M(z) := M_0 + z^{-1}M_1$ is a linear material law, which satisfies (1.7).

Proof. First, M is obviously a linear material law. We set $\varrho_0 := \frac{1}{c_0} \left(\frac{c_1}{2} + \|M_1\|^2 \frac{2}{c_1} \right)$ and for $y \in H$ we use the decomposition $y = x + x'$ with $x \in \overline{R(M_0)}$, $x' \in N(M_0) = R(M_0)^\perp$. We note that the strict accretivity of M_0 extends to $\overline{R(M_0)}$. Then we get for $z = it + \varrho$ with $t \in \mathbb{R}$ and $\varrho > \varrho_0$

$$\begin{aligned} \operatorname{Re} \langle zM(z)y | y \rangle_H &= \operatorname{Re} \langle (it + \varrho)M_0 x | x \rangle_H + \operatorname{Re} \langle M_1 x | x' \rangle_H + \operatorname{Re} \langle M_1 x' | x \rangle_H + \operatorname{Re} \langle M_1 x' | x' \rangle_H \\ &\geq \varrho c_0 |x|_H^2 - 2\|M_1\| |x|_H |x'|_H + c_1 |x'|_H^2 \\ &\geq \left(\varrho c_0 - \|M_1\|^2 \frac{2}{c_1} \right) |x|_H^2 + \frac{c_1}{2} |x'|_H^2 \\ &\geq \frac{c_1}{2} |y|_H^2, \end{aligned}$$

which shows the assertion. \square

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1.3. Examples

This section is devoted to several examples of partial differential equations, which fit into the scheme of evolutionary problems, introduced in the previous section. We start with some classical partial differential equations from mathematical physics. Then we shortly discuss a class of delay differential equations and we conclude this section by studying integro-differential equations with operator-valued convolution kernels.

1.3.1. Classical equations from mathematical physics

Before we can start to give some concrete examples of equations from mathematical physics, we need to introduce some differential operators, which will be used throughout the text.

Definition. Let $\Omega \subseteq \mathbb{R}^n$ open. We define the operators $\text{grad}_0, \text{div}_0$ as the closures of the operators

$$\begin{aligned} \text{grad} |_{C_c^\infty(\Omega)} : C_c^\infty(\Omega) &\subseteq L_2(\Omega) \rightarrow L_2(\Omega)^n \\ \varphi &\mapsto (\partial_i \varphi)_{i \in \{1, \dots, n\}} \end{aligned}$$

and

$$\begin{aligned} \text{div} |_{C_c^\infty(\Omega)^n} : C_c^\infty(\Omega)^n &\subseteq L_2(\Omega)^n \rightarrow L_2(\Omega) \\ (\varphi_i)_{i \in \{1, \dots, n\}} &\mapsto \sum_{i=1}^n \partial_i \varphi_i, \end{aligned}$$

respectively. Similarly, we define the operators $\text{Grad}_0, \text{Div}_0$ as the closures of

$$\begin{aligned} \text{Grad} |_{C_c^\infty(\Omega)^n} : C_c^\infty(\Omega)^n &\subseteq L_2(\Omega)^n \rightarrow L_{2, \text{sym}}(\Omega)^{n \times n} \\ (\varphi_i)_{i \in \{1, \dots, n\}} &\mapsto \left(\frac{1}{2} (\partial_j \varphi_i + \partial_i \varphi_j) \right)_{i, j \in \{1, \dots, n\}} \end{aligned}$$

and

$$\begin{aligned} \text{Div} |_{C_{c, \text{sym}}^\infty(\Omega)^{n \times n}} : C_{c, \text{sym}}^\infty(\Omega)^{n \times n} &\subseteq L_{2, \text{sym}}(\Omega)^{n \times n} \rightarrow L_2(\Omega)^n \\ (\varphi_{ij})_{i, j \in \{1, \dots, n\}} &\mapsto \left(\sum_{j=1}^n \partial_j \varphi_{ij} \right)_{i \in \{1, \dots, n\}}, \end{aligned}$$

respectively. Here, $L_{2, \text{sym}}(\Omega)^{n \times n} := \{f \in L_2(\Omega)^{n \times n} ; f(x)^T = f(x) \quad (x \in \Omega \text{ a.e.})\}$ endowed with the inner product

$$\langle f | g \rangle_{L_{2, \text{sym}}(\Omega)^{n \times n}} := \int_{\Omega} \text{trace}(f(x)^* g(x)) \, dx \quad (f, g \in L_{2, \text{sym}}(\Omega)^{n \times n})$$

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and $C_{c,\text{sym}}^\infty(\Omega)^{n \times n} := C_c^\infty(\Omega)^{n \times n} \cap L_{2,\text{sym}}(\Omega)^{n \times n}$. Then, by integration by parts one gets

$$\begin{aligned}\text{grad}_0 &\subseteq -\text{div}_0^* =: \text{grad}, \\ \text{div}_0 &\subseteq -\text{grad}_0^* =: \text{div}, \\ \text{Grad}_0 &\subseteq -\text{Div}_0^* =: \text{Grad}, \\ \text{Div}_0 &\subseteq -\text{Grad}_0^* =: \text{Div}.\end{aligned}$$

Finally, if $n = 3$ we define curl_0 as the closure of

$$\begin{aligned}\text{curl}|_{C_c^\infty(\Omega)^3} : C_c^\infty(\Omega)^3 &\subseteq L_2(\Omega)^3 \rightarrow L_2(\Omega)^3 \\ (\varphi_i)_{i \in \{1,2,3\}} &\mapsto \begin{pmatrix} 0 & -\partial_3 & \partial_2 \\ \partial_3 & 0 & -\partial_1 \\ -\partial_2 & \partial_1 & 0 \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix}.\end{aligned}$$

Then, again by integration by parts, we have that

$$\text{curl}_0 \subseteq \text{curl}_0^* =: \text{curl}.$$

Remark 1.3.1. By the definitions above, the domain of grad_0 coincides with the classical Sobolev space $H_0^1(\Omega)$, while $D(\text{grad})$ is given by $H^1(\Omega)$. Hence, the elements in the domains of the differential operators indexed with 0 satisfy an additional boundary condition, if the boundary of Ω is smooth enough. These conditions are given as follows:

$$\begin{aligned}u \in D(\text{grad}_0) &\Rightarrow u = 0 \text{ on } \partial\Omega \\ u \in D(\text{div}_0) &\Rightarrow u \cdot \nu = 0 \text{ on } \partial\Omega \\ u \in D(\text{Grad}_0) &\Rightarrow u = 0 \text{ on } \partial\Omega \\ u \in D(\text{Div}_0) &\Rightarrow u\nu = 0 \text{ on } \partial\Omega \\ u \in D(\text{curl}_0) &\Rightarrow u \times \nu = 0 \text{ on } \partial\Omega,\end{aligned}$$

where ν denotes the unit outward normal vector field on $\partial\Omega$. However, we will not discuss the case of smooth boundaries and use the domain description as a suitable generalization of those boundary conditions, which has the advantage that we can deal with arbitrary open sets Ω .

The heat equation

Let $\Omega \subseteq \mathbb{R}^3$ open. The classical heat equation consists of two equations. First, the balance of momentum, given by

$$\partial_{0,\varrho}\vartheta + \text{div } q = f.$$

Here, $\vartheta \in H_\varrho(\mathbb{R}; L_2(\Omega))$ describes the heat density of the medium Ω , $q \in H_\varrho(\mathbb{R}; L_2(\Omega)^3)$ stands for the heat flux, and $f \in H_\varrho(\mathbb{R}; L_2(\Omega))$ is an external heat source. The equation is completed by Fourier's law, given by

$$q = -k \text{grad } \vartheta,$$

where $k : L_2(\Omega)^3 \rightarrow L_2(\Omega)^3$ is a bounded, strictly accretive operator, modelling the heat conductivity of the underlying medium Ω . As k is strictly accretive and bounded, so is k^{-1}

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by Lemma 1.2.16. Thus, we may rewrite the two equations as a system of the form

$$\left(\partial_{0,\varrho} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & k^{-1} \end{pmatrix} + \begin{pmatrix} 0 & \operatorname{div} \\ \operatorname{grad} & 0 \end{pmatrix} \right) \begin{pmatrix} \vartheta \\ q \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}.$$

If we now impose some boundary conditions, say homogeneous Dirichlet boundary conditions, the operator $\begin{pmatrix} 0 & \operatorname{div} \\ \operatorname{grad} & 0 \end{pmatrix}$ will be replaced by $\begin{pmatrix} 0 & \operatorname{div} \\ \operatorname{grad}_0 & 0 \end{pmatrix}$, which is a skew-selfadjoint and hence, m -accretive operator. Thus, we are in the situation of Proposition 1.2.18 with a material law as in Proposition 1.2.20.

The wave equation

Similar to the heat equation, we can provide a formulation of the wave equation

$$\partial_{0,\varrho}^2 u - \Delta u = f$$

within the framework of evolutionary equations. We define $v := \partial_{0,\varrho} u$ and $q := -\operatorname{grad} u$ and obtain a first order formulation of the form

$$\left(\partial_{0,\varrho} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & \operatorname{div} \\ \operatorname{grad} & 0 \end{pmatrix} \right) \begin{pmatrix} v \\ q \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}.$$

Again, if we choose suitable boundary conditions, yielding an m -accretive realization of the operator $\begin{pmatrix} 0 & \operatorname{div} \\ \operatorname{grad} & 0 \end{pmatrix}$, we obtain an evolutionary problem considered in Proposition 1.2.18 with a material law of the form considered in Proposition 1.2.20.

Maxwell's equation

Maxwell's equations of electro-magnetism consist of two equations linking the electric field $E \in H_\varrho(\mathbb{R}; L_2(\Omega)^3)$ and the magnetic field $H \in H_\varrho(\mathbb{R}; L_2(\Omega)^3)$ in an open domain $\Omega \subseteq \mathbb{R}^3$ in the following way

$$\begin{aligned} \partial_{0,\varrho} \varepsilon E + \sigma E - \operatorname{curl} H &= f, \\ \partial_{0,\varrho} \mu H + \operatorname{curl}_0 E &= 0, \end{aligned}$$

where $\varepsilon, \mu \in L(L_2(\Omega)^3)$ are selfadjoint and model the electric permittivity and the magnetic permeability of the medium Ω , respectively. Moreover, $\sigma \in L(L_2(\Omega)^3)$ stands for the conductivity of Ω and $f \in H_\varrho(\mathbb{R}; H)$ is an external current. The first equation results from Ampere's law combined with Ohm's law, while the second equation is Faraday's law. Writing the two equations as a system, we end up with

$$\left(\partial_{0,\varrho} \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} + \begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -\operatorname{curl} \\ \operatorname{curl}_0 & 0 \end{pmatrix} \right) \begin{pmatrix} E \\ H \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix},$$

which is again of the form studied in Proposition 1.2.18 with a material law as in Proposition 1.2.20. Hence, the well-posedness of the problem follows if μ is strictly accretive, ε is strictly

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accretive on its range and σ is strictly accretive on the kernel of ε . In particular, we can allow certain regions where the electric permittivity vanishes which provides a way to deal with the so-called eddy-current approximation (see e.g. [AV10, PP16]).

The equations of visco-elasticity

We denote by $u \in H_\varrho(\mathbb{R}; L_2(\Omega))$ the displacement of an elastic body $\Omega \subseteq \mathbb{R}^3$ and by $\sigma \in H_\varrho(\mathbb{R}; L_{2,\text{sym}}(\Omega)^{3 \times 3})$ the stress tensor. Then, the balance of momentum yields

$$\tilde{\varrho} \partial_{0,\varrho}^2 u - \text{Div } \sigma = f,$$

where $\tilde{\varrho} \in L(L_2(\Omega)^3)$ selfadjoint and strictly accretive is the density of the medium and $f \in H_\varrho(\mathbb{R}; H)$ is an external source term. The equation is completed by a constitutive relation. In visco-elasticity a common relation is given by the Kelvin-Voigt model

$$(\partial_{0,\varrho} C + D) \text{Grad } u = \sigma, \tag{1.8}$$

with $C, D \in L(L_{2,\text{sym}}(\Omega)^{3 \times 3})$ modeling the viscosity and the modulus of elasticity, respectively. Defining $v := \partial_{0,\varrho} u$ as new unknown, the latter equation can be written as

$$(C + \partial_{0,\varrho}^{-1} D) \text{Grad } v = \sigma.$$

If we assume that C is strictly accretive, we obtain the bounded invertibility of $C + \partial_{0,\varrho}^{-1} D$ if we choose $\varrho > 0$ large enough. Indeed

$$C + \partial_{0,\varrho}^{-1} D = C(1 + \partial_{0,\varrho}^{-1} C^{-1} D),$$

and since $\|\partial_{0,\varrho}^{-1} C^{-1} D\| \leq \frac{1}{\varrho} \|C^{-1} D\|$, we obtain the bounded invertibility for $\varrho > 0$ large enough by the Neumann series. Thus, we can write the equations of visco-elasticity in the following way

$$\left(\partial_{0,\varrho} \begin{pmatrix} \tilde{\varrho} & 0 \\ 0 & \partial_{0,\varrho}^{-1} (C + \partial_{0,\varrho}^{-1} D)^{-1} \end{pmatrix} + \begin{pmatrix} 0 & \text{Div} \\ \text{Grad} & 0 \end{pmatrix} \right) \begin{pmatrix} v \\ \sigma \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}.$$

If we choose suitable boundary conditions, say for simplicity homogeneous Neumann boundary conditions for σ , we obtain an evolutionary equation of the form considered in Proposition 1.2.18 with

$$M(z) = \begin{pmatrix} \tilde{\varrho} & 0 \\ 0 & z^{-1} (C + z^{-1} D)^{-1} \end{pmatrix}, \quad A = \begin{pmatrix} 0 & \text{Div}_0 \\ \text{Grad} & 0 \end{pmatrix}.$$

Moreover, the material law M satisfies the well-posedness condition (1.7) of Proposition 1.2.18, since $\tilde{\varrho}$ is assumed to be selfadjoint and strictly accretive, C^{-1} is strictly accretive and

$$\text{Re} \langle (C + z^{-1} D)^{-1} x | x \rangle_{L_{2,\text{sym}}(\Omega)^{3 \times 3}} \geq \text{Re} \langle C^{-1} x | x \rangle_{L_{2,\text{sym}}(\Omega)^{3 \times 3}} - \frac{\frac{1}{\varrho} \|C^{-1}\|^2 \|D\|}{1 - \frac{1}{\varrho} \|C^{-1}\| \|D\|} |x|_{L_{2,\text{sym}}(\Omega)^{3 \times 3}}^2$$

for $z \in \mathbb{C}_{\text{Re} > \varrho}$, $x \in L_{2,\text{sym}}(\Omega)^{3 \times 3}$, yielding that $(C + z^{-1} D)^{-1}$ is strictly accretive uniformly in $z \in \mathbb{C}_{\text{Re} > \varrho}$ for ϱ large enough (note that the last summand tends to 0 as ϱ tends to infinity).

Remark 1.3.2. We note that besides the Kelvin-Voigt model there exist other models for visco-

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elasticity. For instance, a common model uses convolution terms in (1.8) (see e.g. [Daf70a, Daf70b, Tro15c]). More recently, fractional derivatives were used to model elasticity and we refer to [Pod99, Wau14b] for that topic.

The equations of poro-elastic deformation

To illustrate how systems of coupled partial differential equations can be written as evolutionary equations, we treat the equations of poro-elastic deformations, where a diffusion equation is coupled with the equations of linear elasticity. We discuss the equations of poro-elasticity as they were proposed in [MC96] and mathematically studied in [Sho00, MP10] given by

$$\tilde{\varrho} \partial_{0,\varrho}^2 u - \text{grad } \partial_{0,\varrho} \lambda \text{div } u - \text{Div } C \text{Grad } u + \text{grad } \alpha^* p = f, \quad (1.9)$$

$$\partial_{0,\varrho} (c_0 p + \alpha \text{div } u) - \text{div } k \text{grad } p = g. \quad (1.10)$$

Here, $u \in H_\varrho(\mathbb{R}; L_2(\Omega)^3)$ describes the displacement field of an elastic body $\Omega \subseteq \mathbb{R}^3$ and $p \in H_\varrho(\mathbb{R}; L_2(\Omega))$ is the pressure of a fluid diffusing through Ω . The bounded operators $C \in L(L_{2,\text{sym}}(\Omega)^{3 \times 3})$, $k \in L(L_2(\Omega)^3)$ stand for the elasticity tensor and the hydraulic conductivity of the medium, respectively. The function $\tilde{\varrho} \in L_\infty(\Omega)$ describes the density of the medium and the operator $\alpha \in L(L_2(\Omega))$ generalizes the so-called Biot-Willis constant. Finally, let $c_0, \lambda \in L(L_2(\Omega))$, where c_0 models the porosity of the medium and the compressibility of the fluid. We consider the operator

$$\begin{aligned} \text{trace} : L_{2,\text{sym}}(\Omega)^{3 \times 3} &\rightarrow L_2(\Omega) \\ (\Psi_{ij})_{i,j \in \{1,2,3\}} &\mapsto \sum_{i=1}^3 \Psi_{ii} \end{aligned}$$

and its adjoint given by $\text{trace}^* f = \begin{pmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & f \end{pmatrix}$ for $f \in L_2(\Omega)$. Then, using the relations $\text{trace Grad} \subseteq \text{div}$ and $\text{grad} = \text{Div trace}^*$, we can rewrite (1.9) and (1.10) as

$$\begin{aligned} \tilde{\varrho} \partial_{0,\varrho}^2 u - \text{Div} ((\partial_{0,\varrho} \text{trace}^* \lambda \text{trace} + C) \text{Grad } u - \text{trace}^* \alpha^* p) &= f, \\ \partial_{0,\varrho} (c_0 p + \alpha \text{trace Grad } u) - \text{div } k \text{grad } p &= g. \end{aligned}$$

We define

$$\begin{aligned} v &:= \partial_{0,\varrho} u, \\ T &:= C \text{Grad } u, \\ \omega &:= \lambda \text{trace Grad } v - \alpha^* p, \\ q &:= -k \text{grad } p \end{aligned}$$

as new unknowns, which yields, assuming that λ is continuously invertible,

$$\text{trace Grad } v = \lambda^{-1} \omega + \lambda^{-1} \alpha^* p.$$

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Hence, we end up with the following equations

$$\begin{aligned}\partial_{0,\varrho}\tilde{\varrho}v - \operatorname{Div}(T + \operatorname{trace}^* \omega) &= f, \\ \partial_{0,\varrho}c_0p + \alpha\lambda^{-1}\omega + \alpha\lambda^{-1}\alpha^*p + \operatorname{div} q &= g, \\ \lambda^{-1}\omega + \lambda^{-1}\alpha^*p - \operatorname{trace} \operatorname{Grad} v &= 0, \\ \partial_{0,\varrho}C^{-1}T - \operatorname{Grad} v &= 0, \\ k^{-1}q + \operatorname{grad} p &= 0,\end{aligned}$$

where we assume that C and k are boundedly invertible.

Thus, as a system, the equations of poro-elasticity have the form

$$\begin{pmatrix} \partial_{0,\varrho}M_0 + M_1 + U^* \begin{pmatrix} 0 & 0 & 0 & -\operatorname{Div} & 0 \\ 0 & 0 & 0 & 0 & \operatorname{div} \\ 0 & 0 & 0 & 0 & 0 \\ -\operatorname{Grad} & 0 & 0 & 0 & 0 \\ 0 & \operatorname{grad} & 0 & 0 & 0 \end{pmatrix} U \end{pmatrix} \begin{pmatrix} v \\ p \\ \omega \\ T \\ q \end{pmatrix} = \begin{pmatrix} f \\ g \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

where $U := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \operatorname{trace}^* & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$ and

$$M_0 := \begin{pmatrix} \tilde{\varrho} & 0 & 0 & 0 & 0 \\ 0 & c_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & C^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad M_1 := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha\lambda^{-1}\alpha^* & \alpha\lambda^{-1} & 0 & 0 \\ 0 & \lambda^{-1}\alpha^* & \lambda^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & k^{-1} \end{pmatrix}.$$

By choosing suitable boundary conditions, say homogeneous Dirichlet boundary conditions for v and p , we end up with an evolutionary equation of the form discussed in Proposition 1.2.18 with a material law of the form given in Proposition 1.2.20. Hence, the well-posedness can be derived, by imposing suitable constraints on the coefficients involved in order to satisfy the hypothesis of Proposition 1.2.20 for M_0 and M_1 .

1.3.2. Differential equations with delay

Let H be a Hilbert space, $A : D(A) \subseteq H \rightarrow H$ be m -accretive, $M_0, M_1 \in L(H)$ satisfying the assumptions of Proposition 1.2.20. Moreover, let $(h_k)_{k \in \mathbb{N}}$ be a strictly monotone increasing sequence in $\mathbb{R}_{>0}$ with $\eta := \inf\{|h_{k+1} - h_k|; k \in \mathbb{N}\} > 0$ and $(N_k)_{k \in \mathbb{N}}$ a sequence in $L(H)$ with $\sup_{k \in \mathbb{N}} \|N_k\| < \infty$. We consider a delay equation of the form

$$\left(\partial_{0,\varrho}M_0 + M_1 + \sum_{k \in \mathbb{N}} N_k \tau_{-h_k} + A \right) u = f. \quad (1.11)$$

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This is indeed an evolutionary equation with a linear material law given by

$$M(z) = M_0 + z^{-1}M_1 + z^{-1} \sum_{k \in \mathbb{N}} N_k e^{-h_k z} \quad (z \in \mathbb{C} \setminus \{0\}).$$

We note that the series $\sum_{k \in \mathbb{N}} N_k e^{-h_k z}$ converges absolutely for each $z \in \mathbb{C}_{\text{Re} > 0}$. Indeed, we have that

$$\begin{aligned} \sum_{k \in \mathbb{N}} |e^{-h_k z}| &\leq e^{-h_0 \text{Re } z} + \sum_{k=1}^{\infty} \frac{1}{h_k - h_{k-1}} \int_{h_{k-1}}^{h_k} e^{-\text{Re } z s} \, ds \\ &\leq e^{-h_0 \text{Re } z} + \frac{1}{\eta} \int_{h_0}^{\infty} e^{-\text{Re } z s} \, ds \\ &= \left(1 + \frac{1}{\eta \text{Re } z}\right) e^{-h_0 \text{Re } z}, \end{aligned}$$

and since the sequence $(N_k)_{k \in \mathbb{N}}$ is bounded, we derive the absolute convergence of the series with

$$\left\| \sum_{k \in \mathbb{N}} N_k e^{-h_k z} \right\| \leq \sup_{k \in \mathbb{N}} \|N_k\| \left(1 + \frac{1}{\eta \text{Re } z}\right) e^{-h_0 \text{Re } z} \quad (z \in \mathbb{C}_{\text{Re} > 0}).$$

In particular, the norm of $\sum_{k \in \mathbb{N}} N_k e^{-h_k z}$ tends to 0 as $\text{Re } z \rightarrow \infty$. Moreover, we recall that by Proposition 1.2.20, the material law $N(z) := M_0 + z^{-1}M_1$ satisfies (1.7) on some half plane $\mathbb{C}_{\text{Re} > \varrho}$. Hence, choosing ϱ large enough, we derive that also M satisfies (1.7) and hence, the well-posedness of (1.11) follows from Proposition 1.2.18.

1.3.3. Integro-differential equations

In this subsection we study integro-differential equations. The results presented here are based on [Tro15c]. We focus on equations of the form

$$(\partial_{0,\varrho}(1 + k*) + A) u = f, \tag{1.12}$$

as well as

$$(\partial_{0,\varrho}(1 - k*)^{-1} + A) u = f, \tag{1.13}$$

where in both cases $A : D(A) \subseteq H \rightarrow H$ is an m -accretive operator on some Hilbert space H . The kernel k is a suitable operator-valued function in the space $L_{1,\varrho}(\mathbb{R}_{\geq 0}; L(H))$ defined as follows.

Definition. Let $\varrho \in \mathbb{R}$. We call a function $k : \mathbb{R}_{\geq 0} \rightarrow L(H)$ *admissible*, if k is weakly measurable (i.e., for each $x, y \in H$ the mapping $t \mapsto \langle k(t)x | y \rangle_H$ is measurable) and $t \mapsto \|k(t)\|$ is measurable¹. We define

$$\mathcal{L}_{1,\varrho}(\mathbb{R}_{\geq 0}; L(H)) := \left\{ k : \mathbb{R}_{\geq 0} \rightarrow L(H) ; k \text{ admissible, } \int_0^{\infty} \|k(t)\| e^{-\varrho t} \, dt < \infty \right\}$$

¹We note that the measurability of $t \mapsto \|k(t)\|$ follows from the weak measurability of k , if H is separable.

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as well as

$$L_{1,\varrho}(\mathbb{R}_{\geq 0}; L(H)) := \mathcal{L}_{1,\varrho}(\mathbb{R}_{\geq 0}; L(H)) / \cong,$$

where the relation \cong is the usual equality almost everywhere. We equip $L_{1,\varrho}(\mathbb{R}; L(H))$ with the usual norm defined by

$$|k|_{L_{1,\varrho}} := \int_0^\infty \|k(t)\| e^{-\varrho t} dt \quad (k \in L_{1,\varrho}(\mathbb{R}; L(H))).$$

Remark 1.3.3. We note that $L_{1,\varrho}(\mathbb{R}_{\geq 0}; L(H)) \hookrightarrow L_{1,\mu}(\mathbb{R}_{\geq 0}; L(H))$ for $\varrho \leq \mu$ with $|k|_{L_{1,\mu}} \leq |k|_{L_{1,\varrho}}$ for $k \in L_{1,\varrho}(\mathbb{R}_{\geq 0}; L(H))$.

Lemma 1.3.4. *Let $\varrho_0 \in \mathbb{R}$ and $k \in L_{1,\varrho_0}(\mathbb{R}_{\geq 0}; L(H))$. Let $I \subseteq \mathbb{R}$ a bounded interval and $x \in H$. Then the mapping*

$$H \ni y \mapsto \int_0^\infty \langle k(s)x|y \rangle_H \chi_I(t-s) ds \in \mathbb{C}$$

*is a bounded linear functional for each $t \in \mathbb{R}$ and we define $(k * (\chi_I x))(t) \in H$ as the element satisfying*

$$\langle (k * (\chi_I x))(t) | y \rangle_H = \int_0^\infty \langle k(s)x|y \rangle_H \chi_I(t-s) ds \quad (y \in H).$$

*The so defined mapping $k * (\chi_I x) : \mathbb{R} \rightarrow H$ is continuous and*

$$|(k * \chi_I x)(t)|_H \leq \int_0^\infty \|k(s)\| |\chi_I(t-s)x|_H ds \quad (1.14)$$

for each $t \in \mathbb{R}$.

Proof. For $t \in \mathbb{R}$ and $y \in H$ we have that

$$\left| \int_0^\infty \langle k(s)x|y \rangle_H \chi_I(t-s) ds \right|_H \leq \int_0^\infty \|k(s)\| |\chi_I(t-s)x|_H ds |y|_H$$

which proves that the functional is indeed bounded. The linearity is trivial. Hence, by the Riesz representation theorem there exists a unique element $(k * (\chi_I x))(t) \in H$ with

$$\langle (k * (\chi_I x))(t) | y \rangle_H = \int_0^\infty \langle k(s)x|y \rangle_H \chi_I(t-s) ds$$

for each $y \in H$ and the asserted estimate holds. Moreover, for $t, t' \in \mathbb{R}$ we have

$$|(k * (\chi_I x))(t) - (k * (\chi_I x))(t')|_H = \sup_{y \in H, |y|_H=1} \langle (k * (\chi_I x))(t) - (k * (\chi_I x))(t') | y \rangle_H$$

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$$\begin{aligned}
&= \sup_{y \in H, |y|_H=1} \int_0^\infty \langle k(s)x|y \rangle_H (\chi_I(t-s) - \chi_I(t'-s)) \, ds \\
&\leq \int_0^\infty \|k(s)\| |\chi_I(t-s) - \chi_I(t'-s)| \, ds |x|_H \\
&\rightarrow 0 \quad (t' \rightarrow t),
\end{aligned}$$

by dominated convergence. \square

Lemma 1.3.5. *Let $\varrho_0 \in \mathbb{R}$ and $k \in L_{1,\varrho_0}(\mathbb{R}_{\geq 0}; L(H))$. Then the mapping*

$$\begin{aligned}
k* : \text{Sim}(\mathbb{R}; H) &\subseteq H_{\varrho}(\mathbb{R}; H) \rightarrow H_{\varrho}(\mathbb{R}; H) \\
\sum_{i=1}^n \chi_{I_i} x_i &\mapsto \left(\sum_{i=1}^n k * (\chi_{I_i} x_i) \right),
\end{aligned}$$

where $\text{Sim}(\mathbb{R}; H)$ denotes the space of simple functions with values in H , is well-defined. Moreover, $k*$ extends to a bounded linear operator on $H_{\varrho}(\mathbb{R}; H)$ for each $\varrho \geq \varrho_0$ with $\|k*\|_{L(H_{\varrho})} \leq |k|_{L_{1,\varrho}}$. In particular $\|k*\|_{L(H_{\varrho})} \rightarrow 0$ as $\varrho \rightarrow \infty$.

Proof. By Lemma 1.3.4 we have defined $k*$ for functions $\varphi = \chi_I x$ where $I \subseteq \mathbb{R}$ is a bounded interval and $x \in H$. Thus, for $\varphi \in \text{Sim}(\mathbb{R}; H)$, $k * \varphi$ is a continuous function, in particular, it is measurable. Moreover, choosing pairwise disjoint intervals, we derive from (1.14) that

$$|(k * \varphi)(t)|_H \leq \int_0^\infty \|k(s)\| |\varphi(t-s)|_H \, ds \quad (\varphi \in \text{Sim}(\mathbb{R}; H), t \in \mathbb{R}).$$

Hence, for $\varrho \geq \varrho_0$ and $\varphi \in \text{Sim}(\mathbb{R}; H)$ we have that

$$\begin{aligned}
\int_{\mathbb{R}} |(k * \varphi)(t)|_H^2 e^{-2\varrho t} \, dt &\leq \int_{\mathbb{R}} \left(\int_0^\infty \|k(s)\| |\varphi(t-s)|_H \, ds \right)^2 e^{-2\varrho t} \, dt \\
&\leq \int_{\mathbb{R}} \left(\int_0^\infty \|k(s)\| e^{-\varrho s} \, ds \right) \left(\int_0^\infty \|k(s)\| |\varphi(t-s)|_H^2 e^{\varrho s} \, ds \right) e^{-2\varrho t} \, dt \\
&= |k|_{L_{1,\varrho}} \int_0^\infty \|k(s)\| e^{-\varrho s} \int_{\mathbb{R}} |\varphi(t-s)|_H^2 e^{-2\varrho(t-s)} \, dt \, ds \\
&= |k|_{L_{1,\varrho}}^2 |\varphi|_{\varrho}^2,
\end{aligned}$$

which shows the first assertion. The second assertion follows from $\|k*\|_{L(H_{\varrho})} \leq |k|_{L_{1,\varrho}}$ and $|k|_{L_{1,\varrho}} \rightarrow 0$ for $\varrho \rightarrow \infty$, by monotone convergence. \square

In case of a separable Hilbert space H , we have the usual integral expression for the function $k * f$ as the next lemma shows.

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Lemma 1.3.6. *Let $\varrho_0 \in \mathbb{R}$ and $k \in L_{1,\varrho_0}(\mathbb{R}_{\geq 0}; L(H))$, H separable. Then for $f \in H_{\varrho}(\mathbb{R}; H)$ with $\varrho \geq \varrho_0$ we have that*

$$(k * f)(t) = \int_0^{\infty} k(s)f(t-s) \, ds \quad (t \in \mathbb{R} \text{ a.e.}).$$

Proof. We first prove that

$$\mathbb{R}_{\geq 0} \ni t \mapsto k(t)f(t) \in H$$

is measurable. Indeed, first we note that for all $x, y \in H$ we have that

$$t \mapsto \langle x | k(t)^* y \rangle_H = \langle k(t)x | y \rangle_H$$

is measurable, i.e. $t \mapsto k(t)^* y$ is weakly measurable for each $y \in H$. By the Theorem of Pettis (see Theorem B.7), we infer that $t \mapsto k(t)^* y$ is measurable for each $y \in H$. Hence,

$$t \mapsto \langle k(t)f(t) | y \rangle_H = \langle f(t) | k(t)^* y \rangle_H$$

is measurable and thus, again by Theorem B.7 we derive the measurability of $t \mapsto k(t)f(t)$. In particular, the function

$$\mathbb{R}_{\geq 0} \ni s \mapsto k(s)f(t-s) \in H$$

is measurable for each $t \in \mathbb{R}$. Moreover, $\int_0^{\infty} \|k(s)f(t-s)\|_H \, ds < \infty$ for almost every $t \in \mathbb{R}$, since

$$\begin{aligned} \int_{\mathbb{R}} \left(\int_0^{\infty} \|k(s)f(t-s)\|_H \, ds \right)^2 e^{-2\varrho t} \, dt &\leq \int_{\mathbb{R}} \left(\int_0^{\infty} \|k(s)e^{-\varrho s}\| \|f(t-s)e^{-\varrho(t-s)}\|_H \, ds \right)^2 \, dt \\ &\leq \|k\|_{L_{1,\varrho}}^2 \|f\|_{\varrho}^2 \end{aligned}$$

by Young's inequality. Consider now the function $g : \mathbb{R} \rightarrow H$ defined by

$$g(t) := \begin{cases} \int_0^{\infty} k(s)f(t-s) \, ds, & \text{if } \int_0^{\infty} \|k(s)f(t-s)\|_H \, ds < \infty, \\ 0, & \text{otherwise.} \end{cases}$$

Then g is measurable. Indeed, for $y \in H$ we have that

$$\begin{aligned} \int_{\mathbb{R}} \left| \int_0^{\infty} \langle k(s)f(t-s) | y \rangle_H \, ds \right|^2 e^{-2\varrho t} \, dt &\leq \int_{\mathbb{R}} \left(\int_0^{\infty} \|k(s)f(t-s)\|_H \, ds \right)^2 e^{-2\varrho t} \, dt \|y\|_H^2 \\ &\leq \|k\|_{L_{1,\varrho}}^2 \|y\|_H^2 \|f\|_{\varrho}^2 \end{aligned}$$

and since

$$\mathbb{R}_{\geq 0} \times \mathbb{R} \ni (s, t) \mapsto \langle k(s)f(t-s) | y \rangle_H$$

is measurable, we obtain the measurability of

$$t \mapsto \langle g(t) | y \rangle_H$$

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by Fubini's Theorem. Again by Theorem B.7, the measurability of g follows and by the estimate shown above, we have that

$$\int_{\mathbb{R}} \left| \int_0^{\infty} k(s) f(t-s) \, ds \right|^2 e^{-2\varrho t} \, dt \leq |k|_{L_{1,\varrho}}^2 |f|_{\varrho}^2.$$

Summarizing we have shown that

$$\widetilde{k*} : f \mapsto \left(t \mapsto \int_0^{\infty} k(s) f(t-s) \, ds \right)$$

is a well-defined and bounded operator on $H_{\varrho}(\mathbb{R}; H)$. Let now $I \subseteq \mathbb{R}$ be a bounded interval and $x \in H$. Then

$$\int_0^{\infty} \langle k(s)x | y \rangle_H \chi_I(t-s) \, ds = \left\langle \int_0^{\infty} k(s) \chi_I(t-s) x \, ds \middle| y \right\rangle_H \quad (y \in H)$$

which proves that

$$(k * (\chi_I x))(t) = \int_0^{\infty} k(s) \chi_I(t-s) x \, ds$$

for each $t \in \mathbb{R}$. Consequently, we have for each $\varphi \in \text{Sim}(\mathbb{R}; H)$ that

$$(k * \varphi)(t) = \int_0^{\infty} k(s) \varphi(t-s) \, ds \quad (t \in \mathbb{R}),$$

i.e. $k*$ and $\widetilde{k*}$ coincide on the dense set $\text{Sim}(\mathbb{R}; H)$, which yields the assertion. \square

Definition. Let $\varrho_0 \in \mathbb{R}$ and $k \in L_{1,\varrho_0}(\mathbb{R}_{\geq 0}; L(H))$. Then we define $\widehat{k}(z) \in L(H)$ for $z \in \mathbb{C}_{\text{Re} \geq \varrho_0}$ by

$$\langle x | \widehat{k}(z) y \rangle_H := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}_{\geq 0}} e^{-zs} \langle x | k(s) y \rangle_H \, ds \quad (x, y \in H).$$

Remark 1.3.7. We note that $\widehat{k}(z)$ is well-defined by the Riesz representation theorem. Moreover, the mapping $\mathbb{C}_{\text{Re} > \varrho_0} \ni z \mapsto \widehat{k}(z) \in L(H)$ is bounded by $\frac{1}{\sqrt{2\pi}} |k|_{L_{1,\varrho_0}}$ and analytic according to [HP57, Theorem 3.10.1].

Lemma 1.3.8. *Let $\varrho_0 \in \mathbb{R}$ and $k \in L_{1,\varrho_0}(\mathbb{R}_{\geq 0}; L(H))$. Then, $k* = \sqrt{2\pi} \widehat{k}(\partial_{0,\varrho})$ for each $\varrho > \varrho_0$.*

Proof. Since \widehat{k} is bounded on $\mathbb{C}_{\text{Re} > \varrho_0}$, the operator $\widehat{k}(\partial_{0,\varrho})$ is bounded for each $\varrho > \varrho_0$ as well. Moreover, by Lemma 1.3.5 the operator $k*$ is bounded on $H_{\varrho}(\mathbb{R}; H)$, too. Thus, it suffices to show $k * \varphi = \sqrt{2\pi} \widehat{k}(\partial_{0,\varrho}) \varphi$ for $\varphi = \chi_I y$ for some bounded interval $I \subseteq \mathbb{R}$, $y \in H$ and $\varrho > \varrho_0$. So, let $\varrho > \varrho_0$ and $\varphi = \chi_I y$ for some interval $I \subseteq \mathbb{R}$, $y \in H$. Using that $k * \varphi \in L_{1,\varrho}(\mathbb{R}; H)$ by

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(1.14), we can compute

$$\begin{aligned}
\langle x | \sqrt{2\pi} \widehat{k}(\mathbf{i}t + \varrho) (\mathcal{L}_\varrho \varphi)(t) \rangle_H &= \int_{\mathbb{R}} e^{-(\mathbf{i}t + \varrho)s} \langle x | k(s) (\mathcal{L}_\varrho \varphi)(t) \rangle_H \, ds \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-(\mathbf{i}t + \varrho)(s+r)} \langle x | k(s) \varphi(r) \rangle_H \, dr \, ds \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-(\mathbf{i}t + \varrho)r} \langle x | k(s) y \rangle_H \chi_I(r-s) \, dr \, ds \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-(\mathbf{i}t + \varrho)r} \langle x | (k * \varphi)(r) \rangle_H \, dr \\
&= \langle x | \mathcal{L}_\varrho(k * \varphi)(t) \rangle_H
\end{aligned}$$

for each $x \in H, t \in \mathbb{R}$. That shows the assertion. \square

The latter lemma gives that (1.12) and (1.13) are indeed evolutionary problems with $M(z) = 1 + \sqrt{2\pi} \widehat{k}(z)$ and $M(z) = \left(1 + \sqrt{2\pi} \widehat{k}(z)\right)$, respectively. We now address the well-posedness of the problems (1.12) and (1.13). For doing so, we formulate the following conditions.

Condition 1.3.9. Let $\varrho_0 \in \mathbb{R}$ and $k \in L_{1,\varrho_0}(\mathbb{R}_{\geq 0}; L(H))$. We say that k satisfies the condition (a), (b) and (c), respectively, if

- (a) For almost every $t \in \mathbb{R}$, the operator $k(t)$ is selfadjoint.
- (b) For almost every $t, s \in \mathbb{R}$ we have $k(t)k(s) = k(s)k(t)$.
- (c) There exists $\varrho_1 \geq \varrho_0$ and $d \leq 0$ such that

$$t \operatorname{Im} \langle \widehat{k}(\mathbf{i}t + \varrho_1) x | x \rangle_H \geq d |x|_H^2$$

for each $t \in \mathbb{R}, x \in H$.

We first show, that a kernel k satisfying Condition 1.3.9 (a) and (c), also satisfies a similar inequality like in (c) for all $\varrho > \varrho_1$. The precise statement is as follows.

Lemma 1.3.10. Let $\varrho_0 \in \mathbb{R}$ and $k \in L_{1,\varrho_0}(\mathbb{R}_{\geq 0}; L(H))$ satisfying Condition 1.3.9 (a) and (c). Then for each $t \in \mathbb{R}, \varrho \geq \varrho_1$ (where ϱ_1 is chosen according to Condition 1.3.9 (c)) and $x \in H$ one has

$$t \operatorname{Im} \langle \widehat{k}(\mathbf{i}t + \varrho) x | x \rangle_H \geq 4d |x|_H^2.$$

Proof. The proof is based on the proof presented in [CS03, Lemma 3.4] in case of scalar-valued kernels. For $x \in H$ we define

$$f(t) := \langle k(t)x | x \rangle_H \quad (t \in \mathbb{R}_{\geq 0})$$

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and get $f \in L_{1,\varrho_0}(\mathbb{R}_{\geq 0}; \mathbb{R})$ by the selfadjointness of $k(t)$. Moreover, we get

$$\begin{aligned}\widehat{f}(\mathbf{i}t + \varrho) &= \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-(\mathbf{i}t + \varrho)s} \langle k(s)x|x \rangle_H \, ds \\ &= \langle \widehat{k}(-\mathbf{i}t + \varrho)x|x \rangle_H\end{aligned}$$

for each $t \in \mathbb{R}, \varrho \geq \varrho_1$. Hence, we have to show that $t \operatorname{Im} \widehat{f}(-\mathbf{i}t + \varrho) \geq 4d|x|_H^2$ for each $t \in \mathbb{R}, x \in H$. Moreover, we note that since f is real-valued we have that $\operatorname{Im} \widehat{f}(-\mathbf{i}t + \varrho) = -\operatorname{Im} \widehat{f}(\mathbf{i}t + \varrho)$ and hence, we need to prove $t \operatorname{Im} \widehat{f}(\mathbf{i}t + \varrho) \leq -4d|x|_H^2$. Using that $\operatorname{Im} \widehat{f}$ is a harmonic function, we employ the Poisson formula for the half-plane (see e.g. [SS03, p.149]) and get

$$\begin{aligned}\operatorname{Im} \widehat{f}(\mathbf{i}t + \varrho) &= \frac{1}{\pi} \int_{\mathbb{R}} \frac{\varrho - \varrho_1}{(t-s)^2 + (\varrho - \varrho_1)^2} \operatorname{Im} \widehat{f}(\mathbf{i}s + \varrho_1) \, ds \\ &= \frac{\varrho - \varrho_1}{\pi} \left(\int_0^\infty \frac{1}{(t-s)^2 + (\varrho - \varrho_1)^2} \operatorname{Im} \widehat{f}(\mathbf{i}s + \varrho_1) \, ds + \right. \\ &\quad \left. + \int_0^\infty \frac{1}{(t+s)^2 + (\varrho - \varrho_1)^2} \operatorname{Im} \widehat{f}(-\mathbf{i}s + \varrho_1) \, ds \right) \\ &= \frac{\varrho - \varrho_1}{\pi} \int_0^\infty \frac{4st}{((t-s)^2 + (\varrho - \varrho_1)^2)((t+s)^2 + (\varrho - \varrho_1)^2)} \operatorname{Im} \widehat{f}(\mathbf{i}s + \varrho_1) \, ds,\end{aligned}$$

where we have used $\operatorname{Im} \widehat{f}(z^*) = -\operatorname{Im} \widehat{f}(z)$ for $z \in \mathbb{C}_{\operatorname{Re} \geq \varrho_1}$. Using Condition 1.3.9 (c) we estimate

$$\begin{aligned}t \operatorname{Im} \widehat{f}(\mathbf{i}t + \varrho) &\leq -4t^2 \frac{\varrho - \varrho_1}{\pi} d|x|_H^2 \int_0^\infty \frac{1}{((t-s)^2 + (\varrho - \varrho_1)^2)((t+s)^2 + (\varrho - \varrho_1)^2)} \, ds \\ &\leq -4d|x|_H^2 \frac{\varrho - \varrho_1}{\pi} \int_0^\infty \frac{1}{(t-s)^2 + (\varrho - \varrho_1)^2} \, ds \\ &\leq -4d|x|_H^2.\end{aligned}$$

□

With this result at hand, we are able to prove the well-posedness of the integro-differential equations. In fact, we show that the material laws satisfy (1.7) and thus, the well-posedness follows from Proposition 1.2.18. We start with (1.12).

Proposition 1.3.11. *Let $\varrho_0 \in \mathbb{R}$ and $k \in L_{1,\varrho_0}(\mathbb{R}_{\geq 0}; H)$. Moreover, assume that k satisfies Condition 1.3.9 (a) and (c). Then, the material law M defined by $M(z) := 1 + \sqrt{2\pi} \widehat{k}(z)$ satisfies (1.7) on $\mathbb{C}_{\operatorname{Re} > \varrho}$ for some $\varrho \geq \max\{0, \varrho_0\}$.*

Proof. For $x \in H$ and $t \in \mathbb{R}, \varrho \geq \max\{0, \varrho_1\}$ where ϱ_1 is chosen according to Condition 1.3.9

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(c) we estimate

$$\begin{aligned} \operatorname{Re}\langle (it + \varrho)M(it + \varrho)x|x\rangle_H &= \varrho|x|_H^2 + \sqrt{2\pi} \operatorname{Re}\langle (it + \varrho)\widehat{k}(it + \varrho)x|x\rangle_H \\ &= \varrho|x|_H^2 + \sqrt{2\pi} \left(\varrho \operatorname{Re}\langle \widehat{k}(it + \varrho)x|x\rangle_H + t \operatorname{Im}\langle \widehat{k}(it + \varrho)x|x\rangle_H \right) \\ &\geq \left(\varrho(1 - |k|_{L_{1,\varrho}}) + \sqrt{2\pi}4d \right) |x|_H^2, \end{aligned}$$

where we have used Lemma 1.3.10. Thus, choosing ϱ large enough, the assertion follows, since $|k|_{L_{1,\varrho}} \rightarrow 0$ as $\varrho \rightarrow \infty$ by monotone convergence. \square

To deal with (1.13), we additionally need to impose Condition 1.3.9 (b). We start with the following observation.

Lemma 1.3.12. *Let $\varrho_0 \in \mathbb{R}$ and $k \in L_{1,\varrho_0}(\mathbb{R}_{\geq 0}; H)$ such that $|k|_{L_{1,\varrho_0}} < 1$ and assume that k satisfies Condition 1.3.9 (a) and (b). Let $\varrho > \varrho_0$ and $t \in \mathbb{R}$. Then the operator $|1 - \sqrt{2\pi}\widehat{k}(it + \varrho)|$ is boundedly invertible and for each $x \in H$ we have*

$$\begin{aligned} &\operatorname{Re}\langle (it + \varrho)(1 - \sqrt{2\pi}\widehat{k}(it + \varrho))^{-1}x|x\rangle_H \\ &= \varrho|D(it + \varrho)x|_H^2 - \sqrt{2\pi}\varrho \operatorname{Re}\langle \widehat{k}(-it + \varrho)D(it + \varrho)x|D(it + \varrho)x\rangle_H + \\ &\quad - \sqrt{2\pi}t \operatorname{Im}\langle \widehat{k}(-it + \varrho)D(it + \varrho)x|D(it + \varrho)x\rangle_H, \end{aligned}$$

where $D(it + \varrho) := |1 - \sqrt{2\pi}\widehat{k}(it + \varrho)|^{-1}$.

Proof. We have $\|\sqrt{2\pi}\widehat{k}(it + \varrho)\| \leq \|k\|_{L(H_\varrho(\mathbb{R}; H))} \leq |k|_{L_{1,\varrho_0}} < 1$ and hence, $1 - \sqrt{2\pi}\widehat{k}(it + \varrho)$ is boundedly invertible due to the Neumann series. Moreover, by Condition 1.3.9 (a) we have that $\widehat{k}(it + \varrho)^* = \widehat{k}(-it + \varrho)$ and thus, we have that $(1 - \sqrt{2\pi}\widehat{k}(it + \varrho))^*$ is boundedly invertible, too. The latter gives that $|1 - \sqrt{2\pi}\widehat{k}(it + \varrho)|$ is boundedly invertible. Moreover, Condition 1.3.9 (b) yields that $\widehat{k}(it + \varrho)$ is normal and so is $1 - \sqrt{2\pi}\widehat{k}(it + \varrho)$. This implies that $1 - \sqrt{2\pi}\widehat{k}(it + \varrho)$, $(1 - \sqrt{2\pi}\widehat{k}(it + \varrho))^*$ and $D(it + \varrho)$ pairwise commute. Thus, we have

$$\begin{aligned} (1 - \sqrt{2\pi}\widehat{k}(it + \varrho))^{-1} &= (1 - \sqrt{2\pi}\widehat{k}(it + \varrho))^* D(it + \varrho)^2 \\ &= D(it + \varrho)(1 - \sqrt{2\pi}\widehat{k}(-it + \varrho))D(it + \varrho), \end{aligned}$$

where we have used $\widehat{k}(it + \varrho)^* = \widehat{k}(-it + \varrho)$ again. Hence, we compute

$$\begin{aligned} &\operatorname{Re}\langle (it + \varrho)(1 - \sqrt{2\pi}\widehat{k}(it + \varrho))^{-1}x|x\rangle_H \\ &= \operatorname{Re}\langle (-it + \varrho)\langle (1 - \sqrt{2\pi}\widehat{k}(-it + \varrho))D(it + \varrho)x|D(it + \varrho)x\rangle_H \\ &= \varrho|D(it + \varrho)x|_H^2 - \sqrt{2\pi}\varrho \operatorname{Re}\langle \widehat{k}(-it + \varrho)D(it + \varrho)x|D(it + \varrho)x\rangle_H + \\ &\quad - \sqrt{2\pi}t \operatorname{Im}\langle \widehat{k}(-it + \varrho)D(it + \varrho)x|D(it + \varrho)x\rangle_H \end{aligned}$$

for each $x \in H$. \square

Proposition 1.3.13. *Let $\varrho_0 \in \mathbb{R}$ and $k \in L_{1,\varrho_0}(\mathbb{R}_{\geq 0}; H)$. Then there is $\varrho \geq \varrho_0$, such that $1 - \sqrt{2\pi}\widehat{k}(z)$ is boundedly invertible for each $z \in \mathbb{C}_{\operatorname{Re} > \varrho}$. If k satisfies Condition 1.3.9 (a) -*

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(c), then there is some $\varrho \geq \max\{0, \varrho_0\}$, such that $M(z) := (1 - \sqrt{2\pi}\widehat{k}(z))^{-1}$ satisfies (1.7) on $\mathbb{C}_{\operatorname{Re} > \varrho}$.

Proof. First we choose $\varrho_2 \geq \varrho_1$, where ϱ_1 is chosen according to Condition 1.3.9 (c), such that $|k|_{L_{1,\varrho_2}} < 1$. Hence, by Lemma 1.3.12 we have for each $x \in H, t \in \mathbb{R}$ and $\varrho > \varrho_2$

$$\begin{aligned} \operatorname{Re}\langle (it + \varrho)M(it + \varrho)x|x \rangle_H &= \operatorname{Re}\left\langle (it + \varrho)\left(1 - \sqrt{2\pi}\widehat{k}(-it + \varrho)\right)^{-1}x \middle| x \right\rangle_H \\ &= \varrho|D(it + \varrho)x|_H^2 - \sqrt{2\pi}\varrho \operatorname{Re}\langle \widehat{k}(-it + \varrho)D(it + \varrho)x|D(it + \varrho)x \rangle_H + \\ &\quad - \sqrt{2\pi}t \operatorname{Im}\langle \widehat{k}(-it + \varrho)D(it + \varrho)x|D(it + \varrho)x \rangle_H, \end{aligned}$$

where $D(it + \varrho) := |1 - \sqrt{2\pi}\widehat{k}(it + \varrho)|^{-1}$. Choosing $\varrho > \max\{0, \varrho_2\}$ and using Lemma 1.3.10 we infer that

$$\operatorname{Re}\langle (it + \varrho)M(it + \varrho)x|x \rangle_H \geq \left(\varrho(1 - |k|_{L_{1,\varrho}}) + \sqrt{2\pi}4d\right)|D(it + \varrho)x|_H^2$$

Moreover, we have that

$$|x|_H = |D(it + \varrho)^{-1}D(it + \varrho)x|_H = \left|(1 - \sqrt{2\pi}\widehat{k}(it + \varrho))D(it + \varrho)x\right|_H \leq (1 + |k|_{L_{1,\varrho}})|D(it + \varrho)x|_H,$$

and hence,

$$\operatorname{Re}\langle (it + \varrho)M(it + \varrho)x|x \rangle_H \geq \frac{\varrho(1 - |k|_{L_{1,\varrho}}) + \sqrt{2\pi}4d}{(1 + |k|_{L_{1,\varrho}})^2}|x|_H^2.$$

Taking into account that $|k|_{L_{1,\varrho}} \rightarrow 0$ as $\varrho \rightarrow \infty$, we derive the assertion for large enough ϱ . \square

We conclude this section with some classical examples of kernels, satisfying Condition 1.3.9 (a)-(c).

Example 1.3.14. Let $k : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ measurable such that $\int_0^\infty |k(t)|e^{-\varrho_0 t} dt < \infty$ for some $\varrho_0 \in \mathbb{R}$. Then clearly, k satisfies Condition 1.3.9 (a) and (b).

(a) Assume that k is absolutely continuous with $\int_0^\infty |k'(t)|e^{-\varrho_1 t} dt < \infty$ for some $\varrho_1 \in \mathbb{R}$. Then, k satisfies Condition 1.3.9 (c). Indeed, for $t \in \mathbb{R}, \varrho > \max\{\varrho_0, \varrho_1, 0\}$ we have that

$$\begin{aligned} \sqrt{2\pi}\widehat{k}(it + \varrho) &= \int_0^\infty e^{-(it + \varrho)s} k(s) ds \\ &= \int_0^\infty e^{-(it + \varrho)s} \int_0^s k'(r) dr ds + k(0) \int_0^\infty e^{-(it + \varrho)s} ds \\ &= \frac{1}{it + \varrho} \sqrt{2\pi}\widehat{k}'(it + \varrho) + \frac{1}{it + \varrho} k(0) \end{aligned}$$

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and thus, for $x \in H$ we can estimate

$$\begin{aligned}
t \operatorname{Im} \langle \widehat{k}(\mathbf{i}t + \varrho)x|x \rangle_H &= -t \operatorname{Im} \widehat{k}(\mathbf{i}t + \varrho)|x|_H^2 \\
&\geq -|t| |\widehat{k}(\mathbf{i}t + \varrho)| |x|_H^2 \\
&= -|t| \frac{1}{|\mathbf{i}t + \varrho|} \left(|\widehat{k}'(\mathbf{i}t + \varrho)| + \frac{1}{\sqrt{2\pi}} |k(0)| \right) |x|_H^2 \\
&\geq -\frac{1}{\sqrt{2\pi}} (|k'|_{L^1, \varrho_0} + |k(0)|) |x|_H^2.
\end{aligned}$$

- (b) In [Prü09] the kernel is assumed to be non-negative and non-increasing. In fact, this also yields that k satisfies Condition 1.3.9 (c) with $d = 0$. Indeed, we can even generalize this fact to kernels $k \in L_{1, \varrho_0}(\mathbb{R}_{\geq 0}; L(H))$ satisfying Condition 1.3.9(a) such that

$$\langle k(t)x|x \rangle_H \geq 0$$

and

$$\langle (k(t) - k(s))x|x \rangle_H \leq 0,$$

for each $x \in H$ and almost every $t, s \in \mathbb{R}$ with $s \leq t$. First we note that this implies

$$\langle (e^{-\varrho t} k(t) - e^{-\varrho s} k(s))x|x \rangle_H = e^{-\varrho t} \langle (k(t) - k(s))x|x \rangle_H + (e^{-\varrho t} - e^{-\varrho s}) \langle k(s)x|x \rangle_H \leq 0$$

for each $\varrho \geq 0, x \in H$ and almost every $t, s \in \mathbb{R}$ with $s \leq t$. Hence, for $\varrho > \max\{0, \varrho_0\}, t \in \mathbb{R}_{>0}$ and $x \in H$ we obtain

$$\begin{aligned}
&\operatorname{Im} \langle \widehat{k}(\mathbf{i}t + \varrho)x|x \rangle_H \\
&= \operatorname{Im} \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{\mathbf{i}ts} \langle e^{-\varrho s} k(s)x|x \rangle_H \, ds \\
&= \frac{1}{\sqrt{2\pi}} \int_0^\infty \sin(ts) \langle e^{-\varrho s} k(s)x|x \rangle_H \, ds \\
&= \frac{1}{\sqrt{2\pi}} \sum_{k=0}^\infty \left(\int_{2k\frac{\pi}{t}}^{(2k+1)\frac{\pi}{t}} \sin(ts) \langle e^{-\varrho s} k(s)x|x \rangle_H \, ds + \int_{(2k+1)\frac{\pi}{t}}^{2(k+1)\frac{\pi}{t}} \sin(ts) \langle e^{-\varrho s} k(s)x|x \rangle_H \, ds \right) \\
&= \frac{1}{\sqrt{2\pi}} \sum_{k=0}^\infty \left(\int_{2k\frac{\pi}{t}}^{(2k+1)\frac{\pi}{t}} \sin(ts) \langle e^{-\varrho s} k(s)x|x \rangle_H \, ds + \right. \\
&\quad \left. + \int_{2k\frac{\pi}{t}}^{(2k+1)\frac{\pi}{t}} \sin\left(t\left(s + \frac{\pi}{t}\right)\right) \langle e^{-\varrho(s+\frac{\pi}{t})} k(s + \frac{\pi}{t})x|x \rangle_H \, ds \right) \\
&= \frac{1}{\sqrt{2\pi}} \sum_{k=0}^\infty \int_{2k\frac{\pi}{t}}^{(2k+1)\frac{\pi}{t}} \sin(ts) \langle (e^{-\varrho s} k(s) - e^{-\varrho(s+\frac{\pi}{t})} k(s + \frac{\pi}{t}))x|x \rangle_H \, ds \geq 0,
\end{aligned}$$

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which yields

$$t \operatorname{Im} \langle \widehat{k}(\mathfrak{i}t + \varrho)x | x \rangle_H \geq 0$$

for $t > 0$. For $t = 0$ the inequality holds trivially and for $t < 0$ we use the fact that $\widehat{k}(\mathfrak{i}t + \varrho) = (\widehat{k}(-\mathfrak{i}t + \varrho))^*$ by Condition 1.3.9(a) and hence,

$$t \operatorname{Im} \langle \widehat{k}(\mathfrak{i}t + \varrho)x | x \rangle_H = t \operatorname{Im} \langle x | \widehat{k}(-\mathfrak{i}t + \varrho)x \rangle_H = -t \operatorname{Im} \langle \widehat{k}(-\mathfrak{i}t + \varrho)x | x \rangle_H \geq 0.$$

1.4. Notes

The main idea of evolutionary problems is to look at partial differential equations as operator equations in time and space. So, the crucial point is to realize the temporal derivative as an operator itself. Thus, we are actually dealing with the sum of two unbounded operators $B = \partial_{0,\varrho} M(\partial_{0,\varrho})$ and A . However, since we restrict ourselves to the Hilbert space setting and the operator B has this special form, so that Laplace transform techniques are employable, the conditions on the closability of $B + A$ and its continuous invertibility are rather easy to verify. If we are leaving the Hilbert space context or deal with more general operators, the situation gets much more involved and we refer to the famous paper [dPG75] for that topic.

Moreover, we remark that the well-posedness of an evolutionary problem, as it is defined above, does not imply the existence of a C_0 -semigroup, even in the case of Cauchy problems. Indeed, if $A : D(A) \subseteq H \rightarrow H$ is a closed densely defined linear operator and $M = 1$, then the associated evolutionary problem is well-posed in the sense above, if there is $\varrho_1 \in \mathbb{R}$ such that $\mathbb{C}_{\operatorname{Re} > \varrho_1} \subseteq \varrho(-A)$ and

$$\mathbb{C}_{\operatorname{Re} > \varrho_1} \ni z \mapsto (z + A)^{-1} \in L(H)$$

is bounded. Hence, if $-A$ generates a C_0 -semigroup, then the associated evolutionary problem is well-posed by the Hille-Yosida theorem (cf. [EN00, Theorem 3.8]). However, the converse is false in general. Indeed, if we choose $H := L_2(\mathbb{R}_{>0}) \oplus L_2(\mathbb{R}_{>0})$ and set

$$A := \begin{pmatrix} -\mathfrak{i} \mathfrak{m} & -\mathfrak{i} \mathfrak{m} \\ 0 & -\mathfrak{i} \mathfrak{m} \end{pmatrix}$$

on H with maximal domain, then $\mathbb{C}_{\operatorname{Re} > 0} \subseteq \varrho(-A)$ with

$$(z + A)^{-1} = \begin{pmatrix} (z - \mathfrak{i} \mathfrak{m})^{-1} & \mathfrak{i} \mathfrak{m} (z - \mathfrak{i} \mathfrak{m})^{-2} \\ 0 & (z - \mathfrak{i} \mathfrak{m})^{-1} \end{pmatrix} \quad (z \in \mathbb{C}_{\operatorname{Re} > 0})$$

and thus, the associated evolutionary problem is well-posed. Moreover,

$$(z + A)^{-k} = \begin{pmatrix} (z - \mathfrak{i} \mathfrak{m})^{-k} & \mathfrak{i} \mathfrak{m} k (z - \mathfrak{i} \mathfrak{m})^{-(k+1)} \\ 0 & (z - \mathfrak{i} \mathfrak{m})^{-k} \end{pmatrix} \quad (z \in \mathbb{C}_{\operatorname{Re} > 0})$$

for each $k \in \mathbb{N}$ and hence,

$$k \|\mathfrak{m} (z - \mathfrak{i} \mathfrak{m})^{-(k+1)}\| \leq \|(z + A)^{-k}\| \quad (z \in \mathbb{C}_{\operatorname{Re} > 0}, k \in \mathbb{N}).$$

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If $-A$ would generate a C_0 -semigroup, there would exist $M \geq 1$ and $\omega \geq 0$ such that

$$\|(\lambda + A)^{-k}\| \leq \frac{M}{(\lambda - \omega)^k} \quad (\lambda > \omega, k \in \mathbb{N})$$

and consequently

$$(\lambda - \omega)^k k \|m(\lambda - i m)^{-(k+1)}\| \leq M \quad (\lambda > \omega, k \in \mathbb{N}).$$

Estimating the norm of the multiplication operator from below by

$$\frac{\lambda}{\sqrt{k}} \lambda^{-(k+1)} |1 - i \frac{1}{\sqrt{k}}|^{-(k+1)}$$

we infer that

$$\left(\frac{\lambda - \omega}{\lambda}\right)^k \sqrt{k} \frac{1}{\sqrt{(1 + \frac{1}{k})^{k+1}}} \leq M$$

for each $\lambda > \omega$ and $k \in \mathbb{N}$. Letting $\lambda \rightarrow \infty$, we get $\sqrt{k} \frac{1}{\sqrt{(1 + \frac{1}{k})^{k+1}}} \leq M$ for each $k \in \mathbb{N}$, which yields a contradiction as the left-hand side tends to infinity as $k \rightarrow \infty$. This shows, that $-A$ is not the generator of a C_0 -semigroup. Hence, in the framework of evolutionary problems, the property of being the generator of a C_0 -semigroup is more a regularity property than a well-posedness condition. We will study C_0 -semigroups associated with evolutionary problems in Chapter 3.

Besides the examples treated in Section 1.3, the framework of evolutionary problems was used in the study for a broad class of partial differential equations, in particular for coupled systems. We refer to [MPTW16a, MPTW15] for systems occurring in thermo-elasticity, to [MPTW16b] for thermo-piezo-electricity, to [MP10] for poro-elastic deformations and to [PTW15b] for so-called micro-polar elasticity models. Moreover, we refer to [KPS⁺14, PTW14a] for an approach to delay equations, to [PTW15a] for fractional differential equations and to [PTW13, PTW16a, PTW14b, Tro15b] for an approach to control systems. Moreover, more complicated boundary conditions could be treated within the framework of evolutionary equations, see [Pic12, Tro14a, PSTW16]. Finally, we remark that the general structure of the material law $M(\partial_{0,q})$ allows for the treatment of partial differential equations of mixed type. These are equations which are elliptic in one part of the underlying domain, parabolic in another one and hyperbolic in a third one. In particular, one does not need to impose transmission conditions, as these are automatically satisfied by solutions. There exist several approaches for such mixed type problems beginning with the early work of Friedrichs [Fri58], which introduces a framework which is nowadays usually referred to as a Friedrichs system. Other approaches, even in a Banach space setting, were proposed by da Prato and Grisvard [dPG75], Colli and Favini [CF95, CF96] for parabolic-hyperbolic problems and by Favini and Yagi [FY99]. We also refer to However, all these approaches require certain constraints on the operators involved, like a Hille-Yosida type condition or certain decay rates for their resolvents, which do not have to be satisfied by our material laws in general.

We note that for the examples treated in Section 1.3, other approaches can be found in the literature. For instance, we refer to [Hal71, Web76, BP05, BP01] for semigroup approaches to delay differential equations and to [GLS90, Prü09] for different approaches to integral and

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integro-differential equations.

The framework of evolutionary problems as it is introduced in the previous sections is not restricted to the autonomous case. In fact, in [PTWW13] a non-autonomous version of Proposition 1.2.20 was proved, where the operators M_0, M_1 are replaced by operator-valued functions $M_0, M_1 : \mathbb{R} \rightarrow L(H)$. Hence, the corresponding evolutionary equations takes the form

$$(\partial_{0,\varrho} M_0(m) + M_1(m) + A) u = f, \quad (1.15)$$

where $(M_i(m)u)(t) := M_i(t)u(t)$ for $t \in \mathbb{R}$, which covers a class of non-autonomous problems. Later on, this result was generalized in [Wau15b], where abstract operator equations of the form

$$(\partial_{0,\varrho} \mathcal{M} + \mathcal{N} + \mathcal{A})u = f$$

where considered. Here the operators involved act on $H_\varrho(\mathbb{R}; H)$ and do not need to commute with the translation-operator τ_h . A further generalization in a different direction is the study of so-called evolutionary inclusions, that is, one replaces the evolutionary equation by an inclusion of the form

$$(u, f) \in (\partial_{0,\varrho} M(\partial_{0,\varrho}) + A),$$

where A is no longer an operator, but a so-called maximal monotone relation $A \subseteq H \oplus H$, which in particular does not need to be linear. For the topic of maximal monotone relations and differential inclusions we refer to the monographs [Bre71, HP97, Sho97]. Inclusions of the above form were studied in [Tro12, Tro13a] and in [TW14], where a nonlinear analogue of the non-autonomous problem (1.15) was studied.

2. Exponential stability for evolutionary problems

This chapter is devoted to the topic of exponential stability of evolutionary problems. We start to introduce the notion of exponential stability and prove a useful characterization result. In the second part of this chapter, we focus on a certain class of second-order problems, and discuss the exponential stability for this class, by using a suitable reformulation as a first-order problem. We conclude this chapter by studying several examples. The results of this chapter are based on [Tro13b, Tro14b, Tro15a].

2.1. Exponential stability

There is a well-established definition of exponential stability in the framework of strongly continuous semigroups. It is simply defined as the property that for each initial value the corresponding trajectory, which is a continuous function, should decay with a certain exponential rate. Using the variation of constant formula, this yields that exponentially decaying right-hand sides yield exponentially decaying solutions. However, in the framework of evolutionary equations, as it was introduced in the previous chapter, we cannot expect to have continuous solutions. Indeed, as evolutionary equations also cover elliptic-type problems, the solution can not be more regular in time than the given right-hand side. Thus, we need to introduce an adapted notion of exponential stability. Throughout, let H be a Hilbert space, $A : D(A) \subseteq H \rightarrow H$ a densely defined closed linear operator and $M : D(M) \subseteq \mathbb{C} \rightarrow L(H)$ a linear material law.

Definition. Assume that the evolutionary problem associated with M and A is well-posed. We call the problem *exponentially stable with decay rate* $\nu_0 > 0$, if for all $\varrho > s_0(M, A)$ with $\varrho \in S_M$ and $f \in H_\varrho(\mathbb{R}; H) \cap H_{-\nu}(\mathbb{R}; H)$ for some $0 \leq \nu < \nu_0$, it follows that

$$\left(\overline{\partial_{0,\varrho} M(\partial_{0,\varrho}) + A} \right)^{-1} f \in H_{-\nu}(\mathbb{R}; H).$$

The definition of exponential stability states that the causal solution operator of an evolutionary problem leaves the spaces $H_{-\nu}(\mathbb{R}; H)$ invariant. Thus, the exponential decay of a function is replaced by the condition that the function should belong to an exponentially weighted L_2 -space with a positive weighting factor. We emphasize, that it is not enough to assume that the evolutionary equation is well-posed on the space $H_{-\nu}(\mathbb{R}; H)$ in the sense that there exists a unique solution, which depends continuously on the given right-hand side, since then the causality of the solution operator may not hold as the next simple example will show.

Example 2.1.1. We choose $H = \mathbb{C}$, $A = 0$ and $M(z) = 1$. Then the corresponding evolutionary problem reads as

$$\partial_{0,\varrho} u = f.$$

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This problem is solvable in $H_\varrho(\mathbb{R}; H)$ for each $\varrho \neq 0$. However, the solution operator $\partial_{0,\varrho}^{-1}$ is causal if and only if $\varrho > 0$ (cp. Proposition 1.1.6). And indeed, this problem is not exponentially stable in the above sense, since for $f = \chi_{[0,1]} \in \bigcap_{\mu \in \mathbb{R}} H_\mu(\mathbb{R}; H)$ we have that for $\varrho > s_0(M, A) = 0$

$$u(t) = \left(\partial_{0,\varrho}^{-1} f \right) (t) = \begin{cases} 0 & \text{if } t < 0, \\ t & \text{if } 0 \leq t \leq 1, \\ 1 & \text{if } t > 1 \end{cases}$$

and thus, $u \notin H_\mu(\mathbb{R}; H)$ for any $\mu \leq 0$.

Although the solutions of evolutionary problems are not continuous in general, one easy way to obtain continuity is to deal with more regular right-hand sides. Moreover, it turns out that then the exponential stability in the sense above really yields an exponential decay of the solution, as the next lemma shows.

Lemma 2.1.2. *Let the evolutionary problem associated with M and A be well-posed. Then, if $f \in H_\varrho^1(\mathbb{R}; H)$ for some $\varrho > s_0(M, A)$, $\varrho \in S_M$, we have that $u := \left(\overline{\partial_{0,\varrho} M(\partial_{0,\varrho}) + A} \right)^{-1} f \in H_\varrho^1(\mathbb{R}; H)$ with $\partial_{0,\varrho} u = \left(\overline{\partial_{0,\varrho} M(\partial_{0,\varrho}) + A} \right)^{-1} \partial_{0,\varrho} f$. Moreover, if the evolutionary problem is exponentially stable with decay rate $\nu_0 > 0$ and $f \in H_\varrho^1(\mathbb{R}; H) \cap H_{-\nu}^1(\mathbb{R}; H)$ for some $\varrho > s_0(M, A)$, $\varrho \in S_M$ and $0 \leq \nu < \nu_0$, then*

$$|u(t)|_H e^{\nu t} \rightarrow 0 \quad (t \rightarrow \infty).$$

Proof. Let $\varrho > s_0(M, A)$ and $f \in H_\varrho^1(\mathbb{R}; H)$. By $(\partial_{0,\varrho} M(\partial_{0,\varrho}) + A) \partial_{0,\varrho} \subseteq \partial_{0,\varrho} (\partial_{0,\varrho} M(\partial_{0,\varrho}) + A)$ it follows that $\partial_{0,\varrho}^{-1} \left(\overline{\partial_{0,\varrho} M(\partial_{0,\varrho}) + A} \right)^{-1} = \left(\overline{\partial_{0,\varrho} M(\partial_{0,\varrho}) + A} \right)^{-1} \partial_{0,\varrho}^{-1}$. Thus,

$$u = \left(\overline{\partial_{0,\varrho} M(\partial_{0,\varrho}) + A} \right)^{-1} f = \partial_{0,\varrho}^{-1} \left(\overline{\partial_{0,\varrho} M(\partial_{0,\varrho}) + A} \right)^{-1} \partial_{0,\varrho} f \in H_\varrho^1(\mathbb{R}; H),$$

which shows the first assertion. Assume now that the evolutionary problem is exponentially stable with decay rate $\nu_0 > 0$. Then for $f \in H_\varrho^1(\mathbb{R}; H) \cap H_{-\nu}^1(\mathbb{R}; H)$, we have that

$$u \in H_{-\nu}(\mathbb{R}; H) \cap H_\varrho^1(\mathbb{R}; H).$$

Moreover,

$$\partial_{0,\varrho} u = \left(\overline{\partial_{0,\varrho} M(\partial_{0,\varrho}) + A} \right)^{-1} \partial_{0,\varrho} f \in H_{-\nu}(\mathbb{R}; H).$$

The two conditions imply $u \in H_{-\nu}^1(\mathbb{R}; H)$. Indeed, for $\varphi \in C_c^\infty(\mathbb{R}; H)$ we compute

$$\begin{aligned} \langle u | \partial_{0,-\nu,c} \varphi \rangle_{-\nu} &= \int_{\mathbb{R}} \langle u(t) | \varphi'(t) \rangle_H e^{2\nu t} dt \\ &= \int_{\mathbb{R}} \langle u(t) | \varphi'(t) e^{2(\nu+\varrho)t} \rangle_H e^{-2\varrho t} dt \\ &= \left\langle u \left| \partial_{0,\varrho,c} \left(\varphi e^{2(\nu+\varrho)\cdot} \right) - 2(\nu + \varrho) \varphi e^{2(\nu+\varrho)\cdot} \right. \right\rangle_{\varrho} \end{aligned}$$

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$$\begin{aligned} &= \langle \partial_{0,\varrho}^* u - 2(\nu + \varrho)u | \varphi e^{2(\nu+\varrho)\cdot} \rangle_{\varrho} \\ &= \langle -\partial_{0,\varrho} u - 2\nu u | \varphi \rangle_{-\nu} \end{aligned}$$

which yields $u \in H_{-\nu}^1(\mathbb{R}; H)$ with $\partial_{0,-\nu} u = \partial_{0,\varrho} u$. The assertion now follows from Proposition 1.1.8. \square

We now come to the main result of this section, the characterization of exponential stability for a class of well-posed evolutionary problems.

Theorem 2.1.3. *Let the evolutionary problem associated with M and A be well-posed. Moreover, let $\nu_0 > 0$ be such that $] -\nu_0, 0] \subseteq S_M$ and assume that $\mathbb{C}_{\operatorname{Re} > -\nu} \cap D(M)$ is connected for each $\nu_0 - \varepsilon < \nu < \nu_0$ and each $\varepsilon > 0$. Then the evolutionary problem is exponentially stable with decay rate ν_0 if and only if $s_0(M, A) \leq -\nu_0$.*

Proof. We first show that if $s_0(M, A) \leq -\nu_0$, then the evolutionary problem is exponentially stable with decay rate ν_0 . So let $0 \leq \nu < \nu_0$, $\varrho > s_0(M, A)$ with $\varrho \in S_M$ and $f \in H_{\varrho}(\mathbb{R}; H) \cap H_{-\nu}(\mathbb{R}; H)$. Since $-\nu \in S_M$ and $-\nu > s_0(M, A)$ by assumption, Corollary 1.2.10 yields

$$\left(\overline{\partial_{0,\varrho} M(\partial_{0,\varrho}) + A} \right)^{-1} f = \left(\overline{\partial_{0,-\nu} M(\partial_{0,-\nu}) + A} \right)^{-1} f \in H_{-\nu}(\mathbb{R}; H),$$

which shows the assertion.

Assume now that the evolutionary problem is exponentially stable with decay rate ν_0 . Let $\varepsilon > 0$. We show that $s_0(M, A) \leq -\nu$ for each $\nu_0 - \varepsilon < \nu < \nu_0$, which would yield the assertion. So, let $\nu_0 - \varepsilon < \nu < \nu_0$ and choose $\varrho > \max\{s_0(M, A), 0\}$ such that $\mathbb{C}_{\operatorname{Re} \geq \varrho} \subseteq D(M)$. We consider the operator

$$S : L_2(\mathbb{R}_{\geq 0}; H) \rightarrow L_2(\mathbb{R}_{\geq 0}; H)$$

given by $S := e^{\nu m} \left(\overline{\partial_{0,\varrho} M(\partial_{0,\varrho}) + A} \right)^{-1} (e^{\nu m})^{-1}$. Indeed, this operator is well-defined since for $f \in L_2(\mathbb{R}_{\geq 0}; H)$ we have that $(e^{\nu m})^{-1} f \in H_{-\nu}(\mathbb{R}_{\geq 0}; H) \subseteq H_{\varrho}(\mathbb{R}_{\geq 0}; H)$ and hence, $Sf \in L_2(\mathbb{R}_{\geq 0}; H)$ by exponential stability and causality of $\left(\overline{\partial_{0,\varrho} M(\partial_{0,\varrho}) + A} \right)^{-1}$. Now we show that S is closed. For doing so, let $(f_n)_{n \in \mathbb{N}}$ in $L_2(\mathbb{R}_{\geq 0}; H)$ such that $f_n \rightarrow f$ and $Sf_n \rightarrow g$ in $L_2(\mathbb{R}_{\geq 0}; H)$ for some $f, g \in L_2(\mathbb{R}_{\geq 0}; H)$. We derive that $(e^{\nu m})^{-1} f_n \rightarrow (e^{\nu m})^{-1} f$ in $H_{-\nu}(\mathbb{R}_{\geq 0}; H)$ and consequently in $H_{\varrho}(\mathbb{R}_{\geq 0}; H)$. By continuity of $\left(\overline{\partial_{0,\varrho} M(\partial_{0,\varrho}) + A} \right)^{-1}$ and $e^{\nu m}$ we infer $Sf_n \rightarrow Sf$ in $H_{\varrho+\nu}(\mathbb{R}_{\geq 0}; H)$. However, since $L_2(\mathbb{R}_{\geq 0}; H) \hookrightarrow H_{\varrho+\nu}(\mathbb{R}_{\geq 0}; H)$, it follows that $g = Sf$ and hence, S is closed. Thus, by the closed graph theorem, $S \in L(L_2(\mathbb{R}_{\geq 0}; H))$. Now, we consider the following operator

$$T : C_c^\infty(\mathbb{R}; H) \subseteq L_2(\mathbb{R}; H) \rightarrow L_2(\mathbb{R}; H)$$

again defined by $T := e^{\nu m} \left(\overline{\partial_{0,\varrho} M(\partial_{0,\varrho}) + A} \right)^{-1} (e^{\nu m})^{-1}$. The operator is well-defined, since for $\varphi \in C_c^\infty(\mathbb{R}; H)$ and $h := \inf \operatorname{spt} \varphi$ we get $\tau_{-h} \varphi \in L_2(\mathbb{R}_{\geq 0}; H)$ and $T\varphi = \tau_h S \tau_{-h} \varphi$ by the translation invariance of $e^{\nu m} \left(\overline{\partial_{0,\varrho} M(\partial_{0,\varrho}) + A} \right)^{-1} (e^{\nu m})^{-1}$. The latter yields that T is bounded, since S is bounded. Moreover, T is causal and hence, there is $N : \mathbb{C}_{\operatorname{Re} > 0} \rightarrow L(H)$

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bounded and analytic such that

$$\widehat{Tf}(z) = N(z)\widehat{f}(z) \quad (z \in \mathbb{C}_{\operatorname{Re}>0})$$

for each $f \in L_2(\mathbb{R}_{\geq 0}; H)$ by Theorem 1.2.11. Next, we show that

$$N(z + \nu) = (zM(z) + A)^{-1} \quad (z \in \mathbb{C}_{\operatorname{Re}>\varrho}). \quad (2.1)$$

For doing so, let $x \in H$ and set $f(t) := \sqrt{2\pi}\chi_{\mathbb{R}_{\geq 0}}(t)e^{-t}x$. Then $\widehat{f}(z) = \frac{1}{z+1}x$ for $z \in \mathbb{C}_{\operatorname{Re}>0}$. Thus, we have for $t \in \mathbb{R}, \mu > \varrho + \nu$

$$\begin{aligned} N(it + \mu)x &= (it + \mu + 1)N(it + \mu)\widehat{f}(it + \mu) \\ &= (it + \mu + 1)\widehat{Tf}(it + \mu) \\ &= (it + \mu + 1) \left(\mathcal{L}_\mu e^{\nu m} \left(\overline{\partial_{0,\varrho} M(\partial_{0,\varrho}) + A} \right)^{-1} (e^{\nu m})^{-1} f \right) (t) \\ &= (it + \mu + 1) \left(\mathcal{L}_{\mu-\nu} \left(\overline{\partial_{0,\mu-\nu} M(\partial_{0,\mu-\nu}) + A} \right)^{-1} (e^{\nu m})^{-1} f \right) (t), \end{aligned}$$

where we have used $(e^{\nu m})^{-1} f \in H_{\mu-\nu}(\mathbb{R}; H)$ and Corollary 1.2.10. The latter gives

$$\begin{aligned} N(it + \mu)x &= (it + \mu + 1) ((it + \mu - \nu)M(it + \mu - \nu) + A)^{-1} \left(\mathcal{L}_{\mu-\nu} (e^{\nu m})^{-1} f \right) (t) \\ &= ((it + \mu - \nu)M(it + \mu - \nu) + A)^{-1} x, \end{aligned}$$

which shows (2.1). So far, we have shown that $\mathbb{C}_{\operatorname{Re}>\varrho} \ni z \mapsto (zM(z) + A)^{-1}$ has an analytic and bounded extension on $\mathbb{C}_{\operatorname{Re}>-\nu}$ given by $N(\cdot + \nu)$. Thus, it is left to show that $zM(z) + A$ is boundedly invertible for each $z \in D(M) \cap \mathbb{C}_{\operatorname{Re}>-\nu}$. Note that then $(zM(z) + A)^{-1} = N(z + \nu)$ for each $z \in D(M) \cap \mathbb{C}_{\operatorname{Re}>-\nu}$ by the identity theorem. We define the set

$$\Omega := \{z \in D(M) \cap \mathbb{C}_{\operatorname{Re}>-\nu}; zM(z) + A \text{ is boundedly invertible}\} \subseteq D(M) \cap \mathbb{C}_{\operatorname{Re}>-\nu}.$$

We show that Ω is open. Let $z \in \Omega$ and choose $\delta > 0$ such that $B(z, \delta) \subseteq D(M) \cap \mathbb{C}_{\operatorname{Re}>-\nu}$ and

$$\|(z'M(z') - zM(z))(zM(z) + A)^{-1}\| < 1$$

for each $z' \in B(z, \delta)$. Then, for $z' \in B(z, \delta)$ it follows that

$$\begin{aligned} z'M(z') + A &= z'M(z') - zM(z) + zM(z) + A \\ &= \left((z'M(z') - zM(z))(zM(z) + A)^{-1} + 1 \right) (zM(z) + A) \end{aligned} \quad (2.2)$$

is boundedly invertible by the Neumann series. So, Ω is open. Consider now the component C in Ω containing the point $\varrho + 1$. Since Ω is open, so is C . Moreover, by (2.1) and the identity theorem we have that $N(z + \nu) = (zM(z) + A)^{-1}$ for each $z \in C$. Let now $(z_n)_{n \in \mathbb{N}}$ be a sequence in C with $z_n \rightarrow z \in D(M) \cap \mathbb{C}_{\operatorname{Re}>-\nu}$. Since

$$\sup_{n \in \mathbb{N}} \|(z_n M(z_n) + A)^{-1}\| = \sup_{n \in \mathbb{N}} \|N(z_n + \nu)\| < \infty,$$

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we infer, using that for each $n \in \mathbb{N}$

$$zM(z) + A = \left((zM(z) - z_n M(z_n)) (z_n M(z_n) + A)^{-1} + 1 \right) (z_n M(z_n) + A),$$

that $z \in \Omega$. Since Ω is open, we also get $z \in C$, showing that C is closed. Hence, $C = D(M) \cap \mathbb{C}_{\operatorname{Re} > -\nu}$ and consequently $\Omega = D(M) \cap \mathbb{C}_{\operatorname{Re} > -\nu}$, which completes the proof. \square

Remark 2.1.4. The latter theorem especially applies to the case of evolution equations, i.e. equations of the form

$$(\overline{\partial_{0,\rho}} + A) u = f,$$

where $-A$ is the generator of a strongly-continuous semigroup. Indeed, here $M(z) = 1$ for each $z \in \mathbb{C}$, and so the assumptions in Theorem 2.1.3 on the material law M are trivially satisfied. Hence, exponential stability in the sense defined in this section is equivalent to $s_0(1, A) = s_0(A) < 0$. Using now the Theorem of Gearhart-Prüß (see [Prü84] and Theorem 3.3.7 in this thesis), which states that the growth bound of the associated semigroup equals $s_0(A)$, we derive the exponential stability of the evolution equation in the classical sense. Hence, our notion of exponential stability coincides with the classical ones for evolution equations. We will address the relation between semigroups and general evolutionary problems in the next chapter.

We conclude this section by discussing conditions on the linear material law, which yield exponential stability of the associated evolutionary problem.

Proposition 2.1.5. *Let $A : D(A) \subseteq H \rightarrow H$ be m -accretive and $M : D(M) \subseteq \mathbb{C} \rightarrow L(H)$ be a linear material law. Let $\nu_0 > 0$ such that $\mathbb{C}_{\operatorname{Re} > -\nu_0} \setminus D(M)$ is discrete and*

$$\exists c > 0 \forall z \in D(M) \cap \mathbb{C}_{\operatorname{Re} > -\nu_0}, x \in H : \operatorname{Re} \langle zM(z)x | x \rangle_H \geq c|x|_H^2.$$

Then the evolutionary problem associated with M and A is well-posed and exponentially stable with decay rate ν_0 .

Proof. The well-posedness of the evolutionary problem follows from Proposition 1.2.18. Moreover, the material law M satisfies the assumptions in Theorem 2.1.3, since $\mathbb{C}_{\operatorname{Re} > -\nu_0} \setminus D(M)$ is at most countable and so $] -\nu_0, 0] \subseteq S_M$ as well as $D(M) \cap \mathbb{C}_{\operatorname{Re} > -\nu} = \mathbb{C}_{\operatorname{Re} > -\nu} \setminus (\mathbb{C}_{\operatorname{Re} > -\nu_0} \setminus D(M))$ is connected for each $\nu < \nu_0$. Moreover, for each $z \in D(M) \cap \mathbb{C}_{\operatorname{Re} > -\nu_0}$ the operator $zM(z) + A$ is boundedly invertible with $\|(zM(z) + A)^{-1}\| \leq \frac{1}{c}$ by Proposition 1.2.15 and Proposition 1.2.17. Thus, the mapping

$$D(M) \cap \mathbb{C}_{\operatorname{Re} > -\nu_0} \ni z \mapsto (zM(z) + A)^{-1} \in L(H)$$

is bounded and analytic. Since $\mathbb{C}_{\operatorname{Re} > -\nu_0} \setminus D(M)$ is discrete, we find a bounded and analytic extension to the whole $\mathbb{C}_{\operatorname{Re} > -\nu_0}$ by Riemann's Theorem on removable singularities. Thus, $s_0(M, A) \leq -\nu_0$ and hence, the problem is exponentially stable with decay rate ν_0 by Theorem 2.1.3. \square

Proposition 2.1.6. *Let $A : D(A) \subseteq H \rightarrow H$ be m -accretive, $M : D(M) \subseteq \mathbb{C} \rightarrow L(H)$ be a linear-material law and assume that A is boundedly invertible. Let $\nu_0 > 0$ such that*

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$\mathbb{C}_{\operatorname{Re} > -\nu_0} \setminus D(M)$ is discrete and $\delta > 0$ such that

$$K := \sup_{z \in D(M) \cap B[0, \delta]} \|zM(z)\| < \|A^{-1}\|^{-1}$$

and

$$\exists c > 0 \forall z \in (D(M) \cap \mathbb{C}_{\operatorname{Re} > -\nu_0}) \setminus B[0, \delta], x \in H : \operatorname{Re} \langle zM(z)x | x \rangle_H \geq c|x|_H^2.$$

Then, the evolutionary problem associated with M and A is well-posed and exponentially stable with stability rate ν_0 .

Proof. Again, the well-posedness follows from Proposition 1.2.18. Moreover, for $z \in D(M) \cap B[0, \delta]$ the operator

$$zM(z) + A = (zM(z)A^{-1} + 1)A$$

is boundedly invertible with

$$\|(zM(z) + A)^{-1}\| \leq \frac{\|A^{-1}\|}{1 - K\|A^{-1}\|}$$

by the Neumann series and for $z \in (D(M) \cap \mathbb{C}_{\operatorname{Re} > -\nu_0}) \setminus B[0, \delta]$ the invertibility follows from Proposition 1.2.15 and Proposition 1.2.17 with

$$\|(zM(z) + A)^{-1}\| \leq \frac{1}{c}.$$

Hence, the assertion follows again by Riemann's Theorem for removable singularities and Theorem 2.1.3. \square

2.2. Exponential stability for a class of second order evolutionary problems

In this section, we consider second-order differential equations of the following form

$$(\partial_{0,\varrho}^2 M(\partial_{0,\varrho}) + C^*C)u = f, \tag{2.3}$$

where $C : D(C) \subseteq H_0 \rightarrow H_1$ is a densely defined, closed linear operator between two Hilbert spaces H_0, H_1 , which is assumed to be boundedly invertible, and $M : D(M) \subseteq \mathbb{C} \rightarrow L(H)$ is a linear material law, which is given by

$$M(z) := M_0(z) + z^{-1}M_1(z),$$

where $M_0, M_1 : D(M) \subseteq \mathbb{C} \rightarrow L(H_0)$ are analytic and bounded. We want to reformulate this problem as a first-order evolutionary problem, which allows us to study the exponential stability. For doing so, we choose $d > 0$ and define the new unknowns $v := \partial_{0,\varrho}u + du$ and $q := -Cu$. Then, we obtain

$$\partial_{0,\varrho}q = -C\partial_{0,\varrho}u = -Cv + dCu = -Cv - dq$$

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and

$$\begin{aligned}
\partial_{0,\varrho} M(\partial_{0,\varrho})v &= \partial_{0,\varrho}^2 M(\partial_{0,\varrho})u + d\partial_{0,\varrho} M(\partial_{0,\varrho})u \\
&= f + C^*q + dM_0(\partial_{0,\varrho})\partial_{0,\varrho}u + dM_1(\partial_{0,\varrho})u \\
&= f + C^*q + dM_0(\partial_{0,\varrho})v + d(M_1(\partial_{0,\varrho}) - dM_0(\partial_{0,\varrho}))u \\
&= f + C^*q + dM_0(\partial_{0,\varrho})v - d(M_1(\partial_{0,\varrho}) - dM_0(\partial_{0,\varrho}))C^{-1}q,
\end{aligned}$$

which can be written as a system of the form

$$\begin{aligned}
\left(\partial_{0,\varrho} \begin{pmatrix} M(\partial_{0,\varrho}) & 0 \\ 0 & 1 \end{pmatrix} + d \begin{pmatrix} -M_0(\partial_{0,\varrho}) & (M_1(\partial_{0,\varrho}) - dM_0(\partial_{0,\varrho}))C^{-1} \\ 0 & 1 \end{pmatrix} + \right. \\
\left. + \begin{pmatrix} 0 & -C^* \\ C & 0 \end{pmatrix} \right) \begin{pmatrix} v \\ q \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix} \quad (2.4)
\end{aligned}$$

Thus, we arrive at an evolutionary problem with a d -dependent material law

$$M_d(z) := \begin{pmatrix} M(z) & 0 \\ 0 & 1 \end{pmatrix} + z^{-1}d \begin{pmatrix} -M_0(z) & (M_1(z) - dM_0(z))C^{-1} \\ 0 & 1 \end{pmatrix} \quad (z \in D(M)). \quad (2.5)$$

Our main goal is to derive conditions on the material law M , which yield the exponential stability of (2.4) for some $d > 0$.

Remark 2.2.1. We note that, if (2.4) is exponentially stable and $v, q \in H_{-\nu}(\mathbb{R}; H)$ for some $\nu > 0$, that also $u \in H_{-\nu}^1(\mathbb{R}; H) \hookrightarrow C_{-\nu}(\mathbb{R}; H)$, yielding the exponential stability of the original second-order problem. Indeed, since $u = -C^{-1}q$ we derive $u \in H_{-\nu}(\mathbb{R}; H)$ and hence, $\partial_{0,\varrho}u = v - du \in H_{-\nu}(\mathbb{R}; H)$.

We first discuss, how condition (1.7) for the material law M carries over to an analogous estimate for M_d .

Lemma 2.2.2. *Let $z \in D(M)$ and $c > 0$ such that*

$$\operatorname{Re}\langle zM(z)u|u\rangle_{H_0} \geq c|u|_{H_0}^2 \quad (u \in H_0).$$

Then for each $d > 0$, it follows that

$$\operatorname{Re}\langle zM_d(z)(v, q)|(v, q)\rangle_{H_0 \oplus H_1} \geq \min \left\{ c - dK(d), \frac{3}{4}d + \operatorname{Re} z \right\} |(v, q)|_{H_0 \oplus H_1}^2 \quad (v \in H_0, q \in H_1),$$

where $K(d) := \|M_0\|_\infty + (d\|M_0\|_\infty + \|M_1\|_\infty)^2 \|C^{-1}\|^2$ and M_d is given by (2.5).

Proof. Let $v \in H_0, q \in H_1$. Then we estimate

$$\begin{aligned}
&\operatorname{Re}\langle zM_d(z)(v, q)|(v, q)\rangle_{H_0 \oplus H_1} \\
&= \operatorname{Re}\langle zM(z)v - dM_0(z)v + d(M_1(z) - dM_0(z))C^{-1}q|v\rangle_{H_0} + \operatorname{Re}\langle zq + dq|q\rangle_{H_1} \\
&\geq (c - d\|M_0(z)\|)|v|_{H_0}^2 - d\|(M_1(z) - dM_0(z))C^{-1}\| |q|_{H_1}|v|_{H_0} + (\operatorname{Re} z + d)|q|_{H_1}^2 \\
&\geq \left(c - d\|M_0\|_\infty - \frac{1}{4\varepsilon}d^2(d\|M_0\|_\infty + \|M_1\|_\infty)^2 \|C^{-1}\|^2 \right) |v|_{H_0}^2 + (\operatorname{Re} z + d - \varepsilon)|q|_{H_1}^2,
\end{aligned}$$

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for each $\varepsilon > 0$. If we choose $\varepsilon = \frac{d}{4}$, we obtain the assertion. \square

First, we want to check, whether there is some $d > 0$ such that M_d satisfies the assumptions of Proposition 2.1.5, if M does. We begin with the following proposition.

Proposition 2.2.3. *Let $c > 0$ such that for each $z \in D(M)$ and $u \in H_0$ we have that*

$$\operatorname{Re}\langle zM(z)u|u\rangle_{H_0} \geq c|u|_{H_0}^2.$$

Then, there exist $\tilde{c}, d_0 > 0$ and $\varrho_0 > 0$, such that for each $z \in D(M) \cap \mathbb{C}_{\operatorname{Re} > -\varrho_0}$, $v \in H_0$ and $q \in H_1$

$$\operatorname{Re}\langle zM_{d_0}(z)(v, q)|(v, q)\rangle_{H_0 \oplus H_1} \geq \tilde{c}|(v, q)|_{H_0 \oplus H_1}^2,$$

where M_{d_0} is given by (2.5).

Proof. Let $v \in H_0, q \in H_1$. By Lemma 2.2.2 we have

$$\operatorname{Re}\langle zM_d(z)(v, q)|(v, q)\rangle_{H_0 \oplus H_1} \geq \min \left\{ c - dK(d), \frac{3}{4}d + \operatorname{Re} z \right\} |(v, q)|_{H_0 \oplus H_1}^2$$

for each $z \in D(M)$ and $d > 0$, where $K(d) = \|M_0\|_\infty + (d\|M_0\|_\infty + \|M_1\|_\infty)^2 \|C^{-1}\|^2$. Since $dK(d) \rightarrow 0$ as $d \rightarrow 0$, we may choose $d_0 > 0$ such that $d_0K(d_0) < c$. Moreover, we choose $\varrho_0 < \frac{3}{4}d_0$. Then, the above estimate yields that

$$\operatorname{Re}\langle zM_{d_0}(z)(v, q)|(v, q)\rangle_{H_0 \oplus H_1} \geq \min \left\{ c - d_0K(d_0), \frac{3}{4}d_0 - \varrho_0 \right\} |(v, q)|_{H_0 \oplus H_1}^2$$

for each $z \in D(M) \cap \mathbb{C}_{\operatorname{Re} > -\varrho_0}$, which proves the assertion. \square

Corollary 2.2.4. *Assume that there exist $\nu_0, c > 0$ such that $\mathbb{C}_{\operatorname{Re} > -\nu_0} \setminus D(M)$ is discrete and for each $z \in D(M) \cap \mathbb{C}_{\operatorname{Re} > -\nu_0}$ and $u \in H_0$ we have that*

$$\operatorname{Re}\langle zM(z)u|u\rangle_{H_0} \geq c|u|_{H_0}^2.$$

Then there exists $d > 0$ such that the evolutionary problem given by (2.4) is well-posed and exponentially stable.

Proof. By Proposition 2.2.3 there exist $d, \tilde{c}, \varrho_0 > 0$ such that

$$\operatorname{Re}\langle zM_d(z)(v, q)|(v, q)\rangle_{H_0 \oplus H_1} \geq \tilde{c}|(v, q)|_{H_0 \oplus H_1}^2$$

for each $v \in H_0, q \in H_1$ and $z \in D(M) \cap \mathbb{C}_{\operatorname{Re} > -\min\{\nu_0, \varrho_0\}}$. Then the assertion follows from Proposition 2.1.5. \square

We now come to our second exponential stability result Proposition 2.1.6 and again we want to find out, how the assumptions on M carry over to M_d for some $d > 0$.

Proposition 2.2.5. *We assume that*

$$\forall \delta > 0 \exists \varrho_0, c > 0 \forall z \in D(M) \cap \mathbb{C}_{\operatorname{Re} > -\varrho_0} \setminus B[0, \delta], u \in H_0 : \operatorname{Re}\langle zM(z)u|u\rangle_{H_0} \geq c|u|_{H_0}^2, \quad (2.6)$$

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and $\lim_{z \rightarrow 0} M_1(z) = 0$. Then, there exist $\tilde{c}, d_0, \delta_0 > 0$ and $\tilde{\varrho}_0 > 0$, such that

$$\sup_{z \in D(M) \cap B[0, \delta_0]} \|zM_{d_0}(z)\| < \|A^{-1}\|^{-1},$$

and

$$\forall z \in D(M) \cap \mathbb{C}_{\operatorname{Re} > -\tilde{\varrho}_0} \setminus B[0, \delta_0], v \in H_0, q \in H_1 : \operatorname{Re} \langle zM_{d_0}(z)(v, q) | (v, q) \rangle_{H_0 \oplus H_1} \geq \tilde{c} |(v, q)|_{H_0 \oplus H_1}^2,$$

where $A := \begin{pmatrix} 0 & -C^* \\ C & 0 \end{pmatrix}$ and M_{d_0} is given by (2.5). If, additionally $\mathbb{C}_{\operatorname{Re} > -\nu_0} \setminus D(M)$ is discrete for some $\nu_0 > 0$, we get that the evolutionary problem given by (2.4) is exponentially stable for $d = d_0$.

Proof. Let $d > 0$ and $z \in D(M)$. Then we have

$$\begin{aligned} \|zM_d(z)\| &\leq \max \{ \|zM(z)\|, |z| \} + G(d) \\ &\leq \max \{ |z| \|M_0(z)\| + \|M_1(z)\|, |z| \} + G(d) \end{aligned}$$

where

$$G(d) := \sup_{z \in D(M)} \left\| d \begin{pmatrix} -M_0(z) & (M_1(z) - dM_0(z))C^{-1} \\ 0 & 1 \end{pmatrix} \right\|.$$

Since $G(d) \rightarrow 0$ as $d \rightarrow 0$ and $M_1(z) \rightarrow 0$ as $z \rightarrow 0$, we may choose $d_1 > 0$ and $\delta_0 > 0$ small enough, such that

$$\sup_{z \in D(M) \cap B[0, \delta_0]} \|zM_d(z)\| < \|A^{-1}\|^{-1}$$

for each $0 < d < d_1$. Moreover, by Lemma 2.2.2 we have that

$$\operatorname{Re} \langle zM_d(z)(v, q) | (v, q) \rangle_{H_0 \oplus H_1} \geq \min \left\{ c - dK(d), \frac{3}{4}d + \operatorname{Re} z \right\} |(v, q)|_{H_0 \oplus H_1}^2,$$

for each $z \in D(M) \cap \mathbb{C}_{\operatorname{Re} > -\varrho_0} \setminus B[0, \delta]$, $d > 0$ and $v \in H_0, q \in H_1$, where $c, \varrho_0 > 0$ are chosen according to (2.6) for $\delta = \delta_0$ and $K(d) = \|M_0\|_\infty + (d\|M_0\|_\infty + \|M_1\|_\infty \|C^{-1}\|)^2$. Let now $0 < d_0 < d_1$ such that $d_0 K(d_0) < c$ and $0 < \tilde{\varrho}_0 \leq \varrho_0$ such that $\tilde{\varrho}_0 < \frac{3}{4}d_0$, the latter estimate gives

$$\operatorname{Re} \langle zM_{d_0}(z)(v, q) | (v, q) \rangle_{H_0 \oplus H_1} \geq \min \left\{ c - d_0 K(d_0), \frac{3}{4}d - \tilde{\varrho}_0 \right\} |(v, q)|_{H_0 \oplus H_1}^2$$

for each $z \in D(M) \cap \mathbb{C}_{\operatorname{Re} > -\varrho_0} \setminus B[0, \delta_0]$, $d > 0$. The last assertion is a direct consequence of Proposition 2.1.6. \square

2.3. Examples

In this section we illustrate our previous findings by applying them to concrete examples, which were also studied in the literature but using different methods. We start with a class of abstract parabolic-type equations.

2. Exponential stability for evolutionary problems

2.3.1. Exponential stability for equations of parabolic type

We begin by proving a criterion for exponential stability for a class of parabolic-type problems.

Proposition 2.3.1. *Let H_0, H_1 be Hilbert spaces and $C : D(C) \subseteq H_0 \rightarrow H_1$ densely defined closed linear and boundedly invertible. Moreover, let $M_0 \in L(H_0)$ selfadjoint and strictly positive definite and $M_1 : D(M_1) \subseteq \mathbb{C} \rightarrow L(H_1)$ a bounded linear material law with $\mathbb{C}_{\text{Re} > -\nu_0} \setminus D(M_1)$ discrete for some $\nu_0 > 0$, such that*

$$\exists c > 0 \forall z \in D(M_1), x \in H_1 : \text{Re} \langle M_1(z)x | x \rangle_{H_1} \geq c|x|_{H_1}^2.$$

Then the evolutionary problem associated with

$$M(z) := \begin{pmatrix} M_0 & 0 \\ 0 & 0 \end{pmatrix} + z^{-1} \begin{pmatrix} 0 & 0 \\ 0 & M_1(z) \end{pmatrix} \quad (z \in D(M_1) \setminus \{0\})$$

and

$$A := \begin{pmatrix} 0 & -C^* \\ C & 0 \end{pmatrix}$$

is well-posed and exponentially stable with decay rate ν_1 , where $\nu_1 := \min\{\nu_0, \frac{c}{\|M_1\|_\infty^2 \|M_0\| \|C^{-1}\|^2}\}$.

Proof. The well-posedness follows from Proposition 1.2.18. For showing the exponential stability we apply Theorem 2.1.3. We note that M satisfies the assumptions of Theorem 2.1.3, since $\mathbb{C}_{\text{Re} > -\nu_0} \setminus D(M)$ is discrete. Thus, it suffices to prove $s_0(M, A) \leq -\nu_1$. For doing so, let $z \in D(M) \cap \mathbb{C}_{\text{Re} > -\varrho}$ with $0 < \varrho < \nu_1$. We need to show that the operator

$$z \begin{pmatrix} M_0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & M_1(z) \end{pmatrix} + \begin{pmatrix} 0 & -C^* \\ C & 0 \end{pmatrix}$$

is boundedly invertible and the norm of its inverse is bounded in z . We start by showing that the operator is onto. So, let $f \in H_0, g \in H_1$. We consider the operator

$$z (C^{-1})^* M_0 C^{-1} + M_1(z)^{-1} \in L(H_1).$$

This operator is continuously invertible with

$$\left\| \left(z (C^{-1})^* M_0 C^{-1} + M_1(z)^{-1} \right)^{-1} \right\| \leq \frac{1}{\frac{c}{\|M_1\|_\infty^2} - \varrho \|M_0\| \|C^{-1}\|^2} =: \mu.$$

Indeed, we have

$$\begin{aligned} \text{Re} \langle z (C^{-1})^* M_0 C^{-1} x + M_1(z)^{-1} x | x \rangle_{H_1} &= \text{Re} \langle z M_0 C^{-1} x | C^{-1} x \rangle_{H_0} + \text{Re} \langle M_1(z)^{-1} x | x \rangle_{H_1} \\ &\geq -\varrho \|M_0\| \|C^{-1}\|^2 |x|_{H_1}^2 + \frac{c}{\|M_1(z)\|^2} |x|_{H_1}^2 \\ &\geq \left(\frac{c}{\|M_1\|_\infty^2} - \varrho \|M_0\| \|C^{-1}\|^2 \right) |x|_{H_1}^2 \end{aligned}$$

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for each $x \in H_1$, where we have used Lemma 1.2.16. We define

$$\begin{aligned} u &:= C^{-1} \left(z (C^{-1})^* M_0 C^{-1} + M_1(z)^{-1} \right)^{-1} \left((C^*)^{-1} f + M_1(z)^{-1} g \right), \\ v &:= M_1(z)^{-1} (g - Cu), \end{aligned} \quad (2.7)$$

and obtain the estimates

$$\begin{aligned} |u|_{H_0} &\leq \mu \|C^{-1}\| \left(\|C^{-1}\| \|f\|_{H_0} + \frac{1}{c} |g|_{H_1} \right) \\ |v|_{H_1} &\leq \frac{1}{c} \left(|g|_{H_1} + \mu \left(\|C^{-1}\| \|f\|_{H_0} + \frac{1}{c} |g|_{H_1} \right) \right). \end{aligned} \quad (2.8)$$

We show that

$$\left(z \begin{pmatrix} M_0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & M_1(z) \end{pmatrix} + \begin{pmatrix} 0 & -C^* \\ C & 0 \end{pmatrix} \right) \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}. \quad (2.9)$$

Indeed, we have $u \in D(C)$ by definition and $M_1(z)v + Cu = g$. Moreover,

$$\begin{aligned} g - Cu &= g - \left(z (C^{-1})^* M_0 C^{-1} + M_1(z)^{-1} \right)^{-1} \left((C^{-1})^* f + M_1(z)^{-1} g \right) \\ &= \left(z (C^{-1})^* M_0 C^{-1} + M_1(z)^{-1} \right)^{-1} \left(z (C^{-1})^* M_0 C^{-1} g - (C^{-1})^* f \right) \end{aligned}$$

and hence,

$$\left(z (C^{-1})^* M_0 C^{-1} + M_1(z)^{-1} \right) (g - Cu) = (C^{-1})^* (z M_0 C^{-1} g - f).$$

Thus, we read off that

$$\begin{aligned} v &= M_1(z)^{-1} (g - Cu) \\ &= (C^{-1})^* (z M_0 C^{-1} g - f) - z (C^{-1})^* M_0 C^{-1} (g - Cu) \\ &= (C^{-1})^* (z M_0 u - f), \end{aligned}$$

which shows $v \in D(C^*)$ and $z M_0 u - C^* v = f$. Hence, the operator

$$z \begin{pmatrix} M_0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & M_1(z) \end{pmatrix} + \begin{pmatrix} 0 & -C^* \\ C & 0 \end{pmatrix}$$

is onto.

Moreover, it is one-to-one, since for $(u, v) \in D(C) \times D(C^*)$ with (2.9) it immediately follows that u and v are given by (2.7). Summarizing, we have proved that $z \begin{pmatrix} M_0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & M_1(z) \end{pmatrix} + \begin{pmatrix} 0 & -C^* \\ C & 0 \end{pmatrix}$ is boundedly invertible for each $z \in D(M) \cap \mathbb{C}_{\text{Re} > -\varrho}$ and the norm of the inverse is uniformly bounded by (2.8). Hence, since $\mathbb{C}_{\text{Re} > -\varrho} \setminus D(M)$ is discrete, we derive that

$$D(M) \cap \mathbb{C}_{\text{Re} > -\varrho} \ni z \mapsto \left(z \begin{pmatrix} M_0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & M_1(z) \end{pmatrix} + \begin{pmatrix} 0 & -C^* \\ C & 0 \end{pmatrix} \right)^{-1}$$

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has a holomorphic and bounded extension to $\mathbb{C}_{\operatorname{Re} > -\varrho}$. Hence, $s_0(M, A) \leq -\varrho$ and since $0 < \varrho < \nu_1$ was arbitrary, we infer $s_0(M, A) \leq -\nu_1$. \square

In the forthcoming examples we apply this result to two concrete models for heat conduction.

The classical heat equation

We recall from 1.3.1 the classical heat equation with homogeneous Dirichlet boundary conditions, which can be written as an evolutionary equation of the form

$$\left(\partial_{0,\varrho} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & k^{-1} \end{pmatrix} + \begin{pmatrix} 0 & \operatorname{div} \\ \operatorname{grad}_0 & 0 \end{pmatrix} \right) \begin{pmatrix} \vartheta \\ q \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix},$$

where $k \in L(L_2(\Omega)^3)$ is a strictly accretive operator modeling the heat conductivity of a medium $\Omega \subseteq \mathbb{R}^3$. We assume that grad_0 satisfies the Poincaré inequality, i.e.

$$\exists c > 0 \forall u \in D(\operatorname{grad}_0) : |u|_{L_2(\Omega)} \leq c |\operatorname{grad}_0 u|_{L_2(\Omega)^3}, \quad (2.10)$$

which for instance is satisfied, if Ω is bounded (see e.g. [Bre11, p. 290, Corollary 9.19]). Note that (2.10) especially implies that grad_0 is injective and $R(\operatorname{grad}_0)$ is closed. We define

$$\begin{aligned} \iota_{R(\operatorname{grad}_0)} : R(\operatorname{grad}_0) &\rightarrow L_2(\Omega)^3 \\ f &\mapsto f. \end{aligned}$$

An easy computation then yields that $\iota_{R(\operatorname{grad}_0)} \iota_{R(\operatorname{grad}_0)}^* : L_2(\Omega)^3 \rightarrow L_2(\Omega)^3$ is the orthogonal projector on $R(\operatorname{grad}_0)$. We define $C := \iota_{R(\operatorname{grad}_0)}^* \operatorname{grad}_0 : D(\operatorname{grad}_0) \subseteq L_2(\Omega) \rightarrow R(\operatorname{grad}_0)$ and obtain a boundedly invertible closed operator, due to (2.10). Moreover, we have $C^* = -\operatorname{div} \iota_{R(\operatorname{grad}_0)}$, since $\iota_{R(\operatorname{grad}_0)}^*$ is bounded. Using these operators, we derive from the second line of the heat equation

$$\iota_{R(\operatorname{grad}_0)}^* q = -\iota_{R(\operatorname{grad}_0)}^* k \iota_{R(\operatorname{grad}_0)} C \vartheta.$$

Hence, defining $\tilde{k} := \iota_{R(\operatorname{grad}_0)}^* k \iota_{R(\operatorname{grad}_0)}$ and $\tilde{q} = \iota_{R(\operatorname{grad}_0)}^* q$, we infer $\tilde{k}^{-1} \tilde{q} + C \vartheta = 0$. Moreover, since $R(\operatorname{grad}_0)^\perp = N(\operatorname{div})$, we obtain $\operatorname{div} q = -C^* \tilde{q}$ and hence, we may modify the heat equation by writing

$$\left(\partial_{0,\varrho} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \tilde{k}^{-1} \end{pmatrix} + \begin{pmatrix} 0 & -C^* \\ C & 0 \end{pmatrix} \right) \begin{pmatrix} \vartheta \\ \tilde{q} \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix},$$

which is now of the form discussed in Proposition 2.3.1. Hence, we derive that the heat equation is exponentially stable with decay rate $\frac{c_1 c^2}{\|\tilde{k}^{-1}\|^2}$, where c_1 is the accretivity constant of k . This yields that ϑ and \tilde{q} decay exponentially, if the right-hand side does. We note that we can also allow a non-vanishing source term in the second coordinate in the modified heat equation.

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Heat conduction with an additional delay term

As a slight generalization of [KPR15] we replace Fourier's law by the following expression

$$q = -k \operatorname{grad}_0 \vartheta - \tilde{k} \tau_{-h} \operatorname{grad}_0 \vartheta,$$

for some operators $k, \tilde{k} \in L(L_2(\Omega)^3)$ and some $h > 0$. We assume that

$$\exists d > 0 \forall p \in L_2(\Omega)^3 : \operatorname{Re} \langle kp | p \rangle_{L_2(\Omega)^3} \geq d |p|_{L_2(\Omega)^3}^2,$$

$\|\tilde{k}\| < d$ and that the Poincaré inequality (2.10) is satisfied.

Lemma 2.3.2. *Under the above conditions the operator $k + \tilde{k} e^{-hz}$ is uniformly strictly accretive for each $z \in \mathbb{C}_{\operatorname{Re} > -\varrho_1}$ with $\varrho_1 > \frac{1}{h} \log \frac{\|\tilde{k}\|}{d}$.*

Proof. For $z \in \mathbb{C}$ we estimate

$$\operatorname{Re} \langle (k + \tilde{k} e^{-hz}) p | p \rangle_{L_2(\Omega)^3} \geq \left(d - \|\tilde{k}\| e^{-h \operatorname{Re} z} \right) |p|_{L_2(\Omega)^3}^2$$

for each $p \in L_2(\Omega)^3$. Since $d - \|\tilde{k}\| e^{-h \varrho_1} > 0$, the assertion follows. \square

Using the latter lemma and the operator $\iota_{R(\operatorname{grad}_0)}$ from the previous example, we end up with an evolutionary equation of the form

$$\left(\partial_{0,\varrho} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & M_1(\partial_{0,\varrho}) \end{pmatrix} + \begin{pmatrix} 0 & -C^* \\ C & 0 \end{pmatrix} \right) \begin{pmatrix} \vartheta \\ \tilde{q} \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix},$$

where $M_1(z) := \iota_{R(\operatorname{grad}_0)}^* (k + \tilde{k} e^{-hz})^{-1} \iota_{R(\operatorname{grad}_0)}$ for $z \in \mathbb{C}_{\operatorname{Re} > -\varrho_1}$ with $\varrho_1 > \frac{1}{h} \log \frac{\|\tilde{k}\|}{d}$ and $C := \iota_{R(\operatorname{grad}_0)}^* \operatorname{grad}_0$. Hence, Proposition 2.3.1 is applicable and we derive the exponential stability of the latter evolutionary problem.

Integro-differential equations

We consider the following evolutionary problem

$$(\partial_{0,\varrho} + (1 - k*)A) u = f, \tag{2.11}$$

where $A : D(A) \subseteq H \rightarrow H$ is strictly m -accretive (e.g. the Dirichlet Laplacian on a bounded domain) and $k \in L_{1,-\mu}(\mathbb{R}_{\geq 0}; L(H))$ for some $\mu > 0$ with $|k|_{L_{1,-\mu}} < 1$ (cp. 1.3.3 for the definition). Similar to 1.3.3 we require that k satisfies Condition 1.3.9 (a) and (b) and additionally

(c') There exists $0 < \mu_0 < \mu$ such that

$$t \operatorname{Im} \langle \widehat{k}(it - \mu_0)x | x \rangle_H \geq 0$$

for each $t \in \mathbb{R}, x \in H$.

Remark 2.3.3. Note that condition (c') is Condition 1.3.9 (c) with $d = 0$.

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Since $|k|_{L_{1,\mu}} < 1$ we can employ the Neumann series (note that for each $\varrho > 0$, $\|k * \|_{L(H_\varrho)} \leq |k|_{L_{1,\varrho}} \leq |k|_{L_{1,-\mu}}$ by Lemma 1.3.5) and rewrite (2.11) as

$$(\partial_{0,\varrho}(1 - k*)^{-1} + A)u = (1 - k*)^{-1}f. \quad (2.12)$$

Proposition 2.3.4. *Let $c > 0$ and $A : D(A) \subseteq H \rightarrow H$ such that $A - c$ is m -accretive and $k \in L_{1,-\mu}(\mathbb{R}_{\geq 0}; L(H))$ for some $\mu > 0$ with $|k|_{L_{1,-\mu}} < 1$. If k satisfies Condition 1.3.9 (a) and (b) and condition (c'), then the evolutionary problem associated with $M(z) := \left(1 - \sqrt{2\pi}\widehat{k}(z)\right)^{-1}$ and A is well-posed and exponentially stable with decay rate*

$$\mu_1 := \sup \left\{ 0 \leq \nu \leq \mu_0 ; \nu \left(1 - |k|_{L_{1,-\nu}}\right)^{-1} \leq c \right\} > 0.$$

Proof. Note that the evolutionary problem associated with $N(z) := M(z) + z^{-1}c$ for $z \in \mathbb{C}_{\operatorname{Re} > -\mu} \setminus \{0\}$ and $A - c$ is the same as the evolutionary problem associated with M and A . We want to apply Proposition 2.1.5 for the material law N . For doing so, let $t \in \mathbb{R}$ and $\varrho > -\mu_1$. We note that by Lemma 1.3.10 we have that

$$t \operatorname{Im} \langle \widehat{k}(it + \varrho)x | x \rangle_H \geq 0$$

for each $x \in H$. We define $D(it + \varrho) := |1 - \sqrt{2\pi}\widehat{k}(it + \varrho)|^{-1}$ and estimate by using Lemma 1.3.12

$$\begin{aligned} & \operatorname{Re} \langle (it + \varrho)N(it + \varrho)x | x \rangle_H \\ &= \operatorname{Re} \langle (it + \varrho)M(it + \varrho)x | x \rangle_H + c|x|_H^2 \\ &= \operatorname{Re} \langle (it + \varrho)(1 - \sqrt{2\pi}\widehat{k}(-it + \varrho))D(it + \varrho)x | D(it + \varrho)x \rangle_H + c|x|_H^2 \\ &= \varrho \operatorname{Re} \langle (1 - \sqrt{2\pi}\widehat{k}(-it + \varrho))D(it + \varrho)x | D(it + \varrho)x \rangle_H + \\ & \quad - \sqrt{2\pi}t \operatorname{Im} \langle \widehat{k}(-it + \varrho)D(it + \varrho)x | D(it + \varrho)x \rangle_H + c|x|_H^2 \\ &\geq \varrho \operatorname{Re} \langle (1 - \sqrt{2\pi}\widehat{k}(-it + \varrho))D(it + \varrho)x | D(it + \varrho)x \rangle_H + c|x|_H^2 \end{aligned}$$

for each $x \in H$. If $\varrho \geq 0$, the latter term can be estimated by $c|x|^2$. For $\varrho < 0$, we compute

$$\begin{aligned} \operatorname{Re} \langle (1 - \sqrt{2\pi}\widehat{k}(-it + \varrho))D(it + \varrho)x | D(it + \varrho)x \rangle_H &= \operatorname{Re} \langle (1 - \sqrt{2\pi}\widehat{k}(it + \varrho))^{-1}x | x \rangle_H \\ &\leq \|(1 - \sqrt{2\pi}\widehat{k}(it + \varrho))^{-1}\| |x|_H^2 \\ &= \frac{1}{1 - \sqrt{2\pi}\|\widehat{k}(it + \varrho)\|} |x|_H^2 \\ &\leq \frac{1}{1 - |k|_{L_{1,\varrho}}} |x|_H^2 \\ &\leq \frac{1}{1 - |k|_{L_{1,-\mu_1}}} |x|_H^2 \end{aligned}$$

and thus,

$$\operatorname{Re} \langle (it + \varrho)N(it + \varrho)x | x \rangle_H \geq \left(c + \frac{\varrho}{1 - |k|_{L_{1,-\mu_1}}} \right) |x|_H^2.$$

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Since $c + \frac{\varrho}{1-|k|_{L_1, \mu_1}} > c + \frac{-\mu_1}{1-|k|_{L_1, -\mu_1}} \geq 0$, the assertion follows by Proposition 2.1.5. \square

The latter proposition shows the exponential stability of (2.12). However, this also yields the exponential stability of our original problem (2.11) with the same decay rate, since $(1 - k*)^{-1}f \in H_{-\nu}(\mathbb{R}; H) \cap H_{\varrho}(\mathbb{R}; H)$ for $f \in H_{-\nu}(\mathbb{R}; H) \cap H_{\varrho}(\mathbb{R}; H)$ for $0 < \nu \leq \mu_1$ and $\varrho > 0$ by Theorem 1.2.9.

2.3.2. Exponential stability for equations of hyperbolic type

In this section we study different examples of differential equations of second order with respect to time. We begin with another variant of the heat equation.

Dual phase lag heat conduction

Let $\Omega \subseteq \mathbb{R}^3$ such that Poincaré's inequality (2.10) holds. In the theory of thermodynamics with dual phase lags, we have the usual balance of momentum equation

$$\partial_{0,\varrho}\vartheta + \operatorname{div} q = f,$$

where $\vartheta \in H_{\varrho}(\mathbb{R}; L_2(\Omega))$ is the temperature density and $q \in H_{\varrho}(\mathbb{R}; L_2(\Omega)^3)$ denotes the heat flux, but Fourier's law is replaced by

$$(1 + \tau_q \partial_{0,\varrho} + \frac{1}{2} \tau_q^2 \partial_{0,\varrho}^2) q = -\operatorname{grad}(1 + \tau_{\vartheta} \partial_{0,\varrho}) \vartheta,$$

where $\tau_q, \tau_{\vartheta} > 0$ are the so-called phases (see [Tzo95]). For ϱ large enough, we infer that

$$1 + \tau_q \partial_{0,\varrho} + \frac{1}{2} \tau_q^2 \partial_{0,\varrho}^2 = \partial_{0,\varrho}^2 \left(\partial_{0,\varrho}^{-2} + \tau_q \partial_{0,\varrho}^{-1} + \frac{1}{2} \tau_q \right)$$

is boundedly invertible, due to the Neumann series. Hence, we can rewrite the modified Fourier law as

$$q = -(1 + \tau_q \partial_{0,\varrho} + \frac{1}{2} \tau_q^2 \partial_{0,\varrho}^2)^{-1} (1 + \tau_{\vartheta} \partial_{0,\varrho}) \operatorname{grad} \vartheta.$$

Hence, the balance of momentum equation gives

$$\partial_{0,\varrho} \vartheta - \operatorname{div} (1 + \tau_q \partial_{0,\varrho} + \frac{1}{2} \tau_q^2 \partial_{0,\varrho}^2)^{-1} (1 + \tau_{\vartheta} \partial_{0,\varrho}) \operatorname{grad} \vartheta = f.$$

Assuming that $f \in D(\partial_{0,\varrho})$, the latter equation gives

$$\partial_{0,\varrho} (1 + \tau_q \partial_{0,\varrho} + \frac{1}{2} \tau_q^2 \partial_{0,\varrho}^2) (1 + \tau_{\vartheta} \partial_{0,\varrho})^{-1} \vartheta - \operatorname{div} \operatorname{grad} \vartheta = \tilde{f},$$

where $\tilde{f} = (1 + \tau_q \partial_{0,\varrho} + \frac{1}{2} \tau_q^2 \partial_{0,\varrho}^2) (1 + \tau_{\vartheta} \partial_{0,\varrho})^{-1} f$. Assuming homogeneous Dirichlet boundary conditions, we end up with the following problem

$$\partial_{0,\varrho}^2 (\partial_{0,\varrho}^{-1} + \tau_q + \frac{1}{2} \tau_q^2 \partial_{0,\varrho}) (1 + \tau_{\vartheta} \partial_{0,\varrho})^{-1} \vartheta - \operatorname{div} \iota_{R(\operatorname{grad}_0)} \iota_{R(\operatorname{grad}_0)}^* \operatorname{grad}_0 \vartheta = \tilde{f}, \quad (2.13)$$

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which is of the form (2.3) with $C := \iota_{R(\text{grad}_0)}^* \text{grad}_0 : D(\text{grad}_0) \subseteq L_2(\Omega) \rightarrow R(\text{grad}_0)$ and $M(z) = \frac{z^{-1} + \tau_q + \frac{1}{2}\tau_q^2 z}{1 + \tau_\vartheta z} = \tau_q \frac{1 + \frac{1}{2}\tau_q z}{1 + \tau_\vartheta z} + z^{-1} \frac{1}{1 + \tau_\vartheta z}$ for $z \in \mathbb{C} \setminus \{0, -\frac{1}{\tau_\vartheta}\}$. Thus, in the framework of Section 2.2 we have that

$$\begin{aligned} M_0(z) &= \tau_q \frac{1 + \frac{1}{2}\tau_q z}{1 + \tau_\vartheta z}, \\ M_1(z) &= \frac{1}{1 + \tau_\vartheta z}, \end{aligned}$$

which are bounded, if we restrict the domain of M to $\mathbb{C}_{\text{Re} > -\frac{1}{\tau_\vartheta} + \varepsilon} \setminus \{0\}$ for some $\varepsilon > 0$. We want to apply Corollary 2.2.4 for showing the exponential stability of the dual-phase lag model. For doing so, we need to check the uniform accretivity of $zM(z)$ on a suitable right half plane.

Lemma 2.3.5. *Assume that $\tau_\vartheta, \tau_q > 0$ such that $\frac{\tau_q}{\tau_\vartheta} < 2$. Then there exists $0 < \nu_0 < \frac{1}{\tau_\vartheta}$ and $c > 0$ such that*

$$\text{Re} \langle zM(z)x|x \rangle_{L_2(\Omega)} \geq c|x|_{L_2(\Omega)}^2$$

for all $x \in L_2(\Omega)$, $z \in \mathbb{C}_{\text{Re} > -\nu_0} \setminus \{0\}$.

Proof. Since $M(z)$ is just the multiplication with a complex number, it suffices to compute $\text{Re } zM(z)$ for $z \in \mathbb{C} \setminus \{0, -\frac{1}{\tau_\vartheta}\}$. Setting $\mu := \frac{\tau_q}{\tau_\vartheta}$ we compute

$$zM(z) = \frac{1 + \tau_q z + \frac{1}{2}\tau_q^2 z^2}{1 + \tau_\vartheta z} = \frac{1}{2}z\tau_q\mu + \mu(1 - \frac{1}{2}\mu) + \frac{1 - \mu(1 - \frac{1}{2}\mu)}{1 + \tau_\vartheta z}$$

and thus,

$$\text{Re } zM(z) = \frac{1}{2}\tau_q\mu \text{Re } z + \mu(1 - \frac{1}{2}\mu) + \frac{(1 - \mu(1 - \frac{1}{2}\mu))(1 + \tau_\vartheta \text{Re } z)}{|1 + \tau_\vartheta z|^2}.$$

We note that by assumption $0 < \mu(1 - \frac{1}{2}\mu) \leq \frac{1}{2}$ and thus,

$$\text{Re } zM(z) \geq \frac{1}{2}\tau_q\mu \text{Re } z + \mu(1 - \frac{1}{2}\mu)$$

for each $z \in \mathbb{C}_{\text{Re} > -\frac{1}{\tau_\vartheta}} \setminus \{0\}$. Hence, for $0 < \nu_0 < \min\{\frac{2-\mu}{\tau_q}, \frac{1}{\tau_\vartheta}\}$ we derive the assertion with $c = \mu(1 - \frac{1}{2}\mu) - \frac{1}{2}\tau_q\mu\nu_0 > 0$. \square

As an immediate consequence of the latter Lemma and Corollary 2.2.4, we derive the following stability result for the dual phase lag model.

Proposition 2.3.6. *Let $\Omega \subseteq \mathbb{R}^3$ open such that Poincaré's inequality (2.10) holds. Let $\tau_q, \tau_\vartheta > 0$ such that $\frac{\tau_q}{\tau_\vartheta} < 2$. Then the second order problem (2.13) is well-posed and exponentially stable.*

Remark 2.3.7. We note that these conditions on τ_q, τ_ϑ coincide with those imposed in [Qui02], where the same exponential stability result is stated. For further results on the well-posedness and asymptotic behaviour for phase lag models in heat conduction we refer to [BQR14, QR08, QR07] and the references therein.

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Abstract damped wave equation

Let H_0, H_1 be Hilbert spaces, $C : D(C) \subseteq H_0 \rightarrow H_1$ a densely defined closed linear operator, which is assumed to be boundedly invertible and $M_0, M_1 \in L(H_0)$ such that M_0 is selfadjoint and accretive and M_1 is strictly accretive. We consider the second order problem of the form

$$(\partial_{0,\varrho}^2 M_0 + \partial_{0,\varrho} M_1 + C^* C) u = f.$$

Obviously, this is a problem of the form (2.3) with $M(z) = M_0 + z^{-1} M_1$ for $z \in \mathbb{C} \setminus \{0\}$, and hence,

$$M_0(z) := M_0, \quad M_1(z) := M_1 \quad (z \in \mathbb{C} \setminus \{0\}).$$

Moreover, we have

$$\operatorname{Re} \langle z M(z) x | x \rangle_{H_0} = \operatorname{Re} z \langle M_0 x | x \rangle_{H_0} + \operatorname{Re} \langle M_1 x | x \rangle_{H_0} \quad (z \in \mathbb{C} \setminus \{0\}, x \in H_0)$$

and hence, for $0 < \nu_0 < \frac{c}{\|M_0\|}$, where $c > 0$ denotes the accretivity constant of M_1 , we have that

$$\operatorname{Re} \langle z M(z) x | x \rangle_{H_0} \geq (c - \nu_0 \|M_0\|) |x|_{H_0}^2 \quad (z \in \mathbb{C}_{\operatorname{Re} > -\nu_0} \setminus \{0\}, x \in H_0),$$

which yields the exponential stability according to Corollary 2.2.4.

A particular case of the latter abstract equation is the damped wave equation. Indeed, let $\Omega \subseteq \mathbb{R}^n$ open, such that the Poincaré inequality (2.10) holds, $H_0 = L_2(\Omega)$, $H_1 = L_2(\Omega)^n$, $M_0 = 1$ and choose $C := \iota_{R(\operatorname{grad}_0)}^* \operatorname{grad}_0$. Then C is invertible and

$$(\partial_{0,\varrho}^2 + \partial_{0,\varrho} M_1 + C^* C) u = (\partial_{0,\varrho}^2 + \partial_{0,\varrho} M_1 - \operatorname{div} \operatorname{grad}_0) u,$$

which is the classical wave equation. However, we emphasize that the abstract equation covers a broader class, since M_0 is allowed to have a non-trivial kernel. In particular, we also recover the heat equation by setting $M_0 = 0$. But also mixed type problems are accessible. Indeed, for instance let $M_1 = 1$ and $M_0 = \chi_{\Omega_0}$ for some open subset $\Omega_0 \subseteq \Omega$. Then the corresponding equation reads as

$$(\partial_{0,\varrho}^2 \chi_{\Omega_0} + \partial_{0,\varrho} - \operatorname{div} \operatorname{grad}_0) u = f,$$

which is a damped wave equation on Ω_0 and the heat equation on $\Omega \setminus \Omega_0$. Note in particular, that we do not need to impose any transmission condition explicitly, as these are encoded in the constraint that u belongs to the domain of $\partial_{0,\varrho}^2 \chi_{\Omega_0} + \partial_{0,\varrho} - \operatorname{div} \operatorname{grad}_0$.

Integro-differential equations

We conclude this section by discussing the exponential stability of integro-differential equations of the form

$$(\partial_{0,\varrho}^2 (1 - k*)^{-1} + C^* C) u = f, \tag{2.14}$$

where $C : D(C) \subseteq H_0 \rightarrow H_1$ is a densely defined closed linear operator between Hilbert spaces H_0, H_1 , which is assumed to be boundedly invertible and $k \in L_{1,-\tilde{\mu}}(\mathbb{R}_{\geq 0}; L(H_0))$ for some $\tilde{\mu} > 0$. We formulate the following conditions for the kernel k :

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Condition 2.3.8.

- (a) There exists $0 < \mu \leq \tilde{\mu}$ such that for all $z \in \mathbb{C}_{\operatorname{Re} \geq -\mu}$ the operator $(1 - \sqrt{2\pi}\widehat{k}(z))$ is boundedly invertible and

$$\mathbb{C}_{\operatorname{Re} \geq -\mu} \ni z \mapsto (1 - \sqrt{2\pi}\widehat{k}(z))^{-1} \in L(H_0)$$

is bounded.

- (b) For each $\varrho_0 > 0$ there is $c_{\varrho_0} > 0$ such that

$$\forall t \in \mathbb{R}, \varrho \geq \varrho_0, x \in H_0 : \operatorname{Re} \langle (it + \varrho)(1 - \sqrt{2\pi}\widehat{k}(it + \varrho))^* x | x \rangle_{H_0} \geq c_{\varrho_0} |x|_{H_0}^2.$$

- (c) For almost every $t \in \mathbb{R}_{\geq 0}$ the operator $k(t)$ is selfadjoint.
(d) For almost every $t, s \in \mathbb{R}_{\geq 0}$ we have $k(t)k(s) = k(s)k(t)$.
(e) For each $\delta > 0$ there exists $0 < \varrho_0 \leq \tilde{\mu}$ and a function $g_\delta : \mathbb{R}_{\geq -\varrho_0} \rightarrow \mathbb{R}_{\geq 0}$ continuous in 0 with $g_\delta(0) > 0$ such that

$$\forall |t| > \delta, \varrho \geq -\varrho_0, x \in H_0 : t \operatorname{Im} \langle \widehat{k}(it + \varrho)x | x \rangle_{H_0} \geq g_\delta(\varrho) |x|_{H_0}^2. \quad (2.15)$$

Remark 2.3.9. In [Tro15a] we did not impose the conditions (a) and (b), but require the somehow stronger condition $|k|_{L_1} < 1$ (see the next lemma). However, since we also want to cover the kernels considered by the authors in [CS11], we use these slightly weaker conditions here.

Lemma 2.3.10. *Let $k \in L_{1, -\tilde{\mu}}(\mathbb{R}_{\geq 0}; L(H))$ for some $\tilde{\mu} > 0$.*

- (a) *If $|k|_{L_1} < 1$ and k satisfies Condition 2.3.8 (e), then k satisfies Condition 2.3.8 (a) and (b).*
(b) *If k satisfies Condition 2.3.8 (c), then (2.15) is equivalent to*

$$\forall t > \delta, \varrho \geq -\varrho_0, x \in H_0 : t \operatorname{Im} \langle \widehat{k}(it + \varrho)x | x \rangle_{H_0} \geq g_\delta(\varrho) |x|_{H_0}^2.$$

- (c) *For each $\varrho > -\tilde{\mu}$ we have that $(t \mapsto tk(t)) \in L_{1, \varrho}(\mathbb{R}_{\geq 0}; L(H_0))$.*

Proof. (a) Let $|k|_{L_1} < 1$ and assume k satisfies Condition 2.3.8 (e). Since the mapping

$$\varrho \mapsto |k|_{L_{1, \varrho}}$$

is continuous, we find $0 < \mu \leq \tilde{\mu}$ such that $|k|_{L_{1, -\mu}} < 1$. Hence, Condition 2.3.8 (a) follows, since for $z \in \mathbb{C}_{\operatorname{Re} \geq -\mu}$ we have $\sqrt{2\pi} \|\widehat{k}(z)\| \leq |k|_{L_{1, -\mu}}$ and thus, the assertion follows by using the Neumann series. Moreover, Condition 2.3.8 (e) gives that for all $x \in H_0, \varrho \geq 0$ and $t \in \mathbb{R}$ we have that

$$t \operatorname{Im} \langle \widehat{k}(it + \varrho)x | x \rangle_{H_0} \geq 0.$$

Indeed, for $t = 0$, this inequality is trivial and for $t \neq 0$, the term on the left-hand side can be estimated from below by $g_{\frac{|t|}{2}}(\varrho) |x|_{H_0}^2 \geq 0$. Let now $\varrho_0 > 0$. Then, for $\varrho \geq \varrho_0, t \in$

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$\mathbb{R}, x \in H_0$ we estimate

$$\begin{aligned} & \operatorname{Re}\langle (it + \varrho)(1 - \sqrt{2\pi}\widehat{k}(it + \varrho)^*)x|x\rangle_{H_0} \\ &= \varrho \operatorname{Re}\langle (1 - \sqrt{2\pi}\widehat{k}(it + \varrho)^*)x|x\rangle_{H_0} - \sqrt{2\pi}t \operatorname{Im}\langle \widehat{k}(it + \varrho)^*x|x\rangle_{H_0} \\ &\geq \varrho(1 - |k|_{L_1})|x|_{H_0}^2 + \sqrt{2\pi}t \operatorname{Im}\langle \widehat{k}(it + \varrho)x|x\rangle_{H_0} \\ &\geq \varrho_0(1 - |k|_{L_1})|x|_{H_0}^2. \end{aligned}$$

(b) Assume that $k(t)$ is selfadjoint for almost every $t \geq 0$. Then $\widehat{k}(-it + \varrho) = \widehat{k}(it + \varrho)^*$ and hence, the assertion follows.

(c) This is clear, since $t e^{-\varrho t} \leq \frac{1}{\varrho + \tilde{\mu}} e^{-1} e^{\tilde{\mu}t}$ for each $t \geq 0$. \square

Before we come to examples of kernels satisfying Condition 2.3.8, we prove the exponential stability of the corresponding second-order problem (2.14).

Proposition 2.3.11. *Let $C : D(C) \subseteq H_0 \rightarrow H_1$ a densely defined closed linear operator between Hilbert spaces H_0, H_1 , which is boundedly invertible and $k \in L_{1, -\tilde{\mu}}(\mathbb{R}_{\geq 0}; L(H_0))$ for some $\tilde{\mu} > 0$. Moreover, assume that k satisfies Condition 2.3.8. Then the evolutionary problem given by (2.14) is well-posed and exponentially stable.*

Proof. We apply Proposition 2.2.5. First, we choose $0 < \mu \leq \tilde{\mu}$ according to (a). Hence, $M(z) := M_0(z) := (1 - \sqrt{2\pi}\widehat{k}(z))^{-1}$ for $z \in \mathbb{C}_{\operatorname{Re} \geq -\mu}$ is well-defined and bounded. Moreover, $\lim_{z \rightarrow 0} M_1(z) = 0$ as $M_1 = 0$. So it suffices to check (2.6). For doing so, let $\delta > 0$. Moreover, let $0 < \varrho_0 \leq \min\{\mu, \frac{\delta}{2}\}$ to be specified later. Then, for $z \in \mathbb{C}_{\operatorname{Re} > -\varrho_0} \setminus B[0, \delta]$ and $x \in H_0$ we estimate, using the representation $z = it + \varrho$ for $\varrho > -\varrho_0, t \in \mathbb{R}$

$$\begin{aligned} \operatorname{Re}\langle (it + \varrho)M(it + \varrho)x|x\rangle_{H_0} &= \operatorname{Re}\langle (it + \varrho)(1 - \sqrt{2\pi}\widehat{k}(it + \varrho))^{-1}x|x\rangle_{H_0} \\ &= \operatorname{Re}\langle (it + \varrho)(1 - \sqrt{2\pi}\widehat{k}(it + \varrho)^*)D(it + \varrho)x|D(it + \varrho)x\rangle_{H_0} \\ &= \operatorname{Re}\langle \varrho(1 - \sqrt{2\pi}\widehat{k}(-it + \varrho))D(it + \varrho)x|D(it + \varrho)x\rangle_{H_0} + \\ &\quad - \sqrt{2\pi}t \operatorname{Im}\langle \widehat{k}(-it + \varrho)D(it + \varrho)x|D(it + \varrho)x\rangle_{H_0} \end{aligned}$$

where $D(it + \varrho) := |1 - \sqrt{2\pi}\widehat{k}(it + \varrho)|^{-1}$. We estimate the latter expression in two steps: first for $\varrho \geq \varrho_0$ and second, for $\varrho \in]-\varrho_0, \varrho_0[$. So let first, $\varrho \geq \varrho_0$. Then, by (b) there is some $c_{\varrho_0} > 0$ such that

$$\operatorname{Re}\langle (it + \varrho)M(it + \varrho)x|x\rangle_{H_0} \geq c_{\varrho_0}|D(it + \varrho)x|_{H_0}^2 \quad (\varrho \geq \varrho_0).$$

For $\varrho \in]-\varrho_0, \varrho_0[$, we infer that $|t| > \delta - \varrho_0 \geq \frac{\delta}{2}$, since $|it + \varrho| > \delta$ by assumption. Hence, we obtain

$$\begin{aligned} \operatorname{Re}\langle (it + \varrho)M(it + \varrho)x|x\rangle_{H_0} &\geq \left(-\varrho(1 + |k|_{L_{1, -\varrho}}) + \sqrt{2\pi}g_{\frac{\delta}{2}}(\varrho)\right)|D(it + \varrho)x|_{H_0}^2 \\ &\geq \left(-\varrho_0(1 + |k|_{L_{1, -\varrho_0}}) + \sqrt{2\pi} \inf_{\nu \in [-\varrho_0, \varrho_0]} g_{\frac{\delta}{2}}(\nu)\right)|D(it + \varrho)x|_{H_0}^2. \end{aligned}$$

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Since $g_{\frac{\delta}{2}}$ is continuous in 0 we infer

$$\left(-\varrho_0(1 + |k|_{L_1, -\varrho_0}) + \sqrt{2\pi} \inf_{\nu \in [-\varrho_0, \varrho_0]} g_{\frac{\delta}{2}}(\nu) \right) \rightarrow \sqrt{2\pi} g_{\frac{\delta}{2}}(0) > 0 \quad (\varrho_0 \rightarrow 0)$$

and thus, we may choose $0 < \varrho_0 \leq \min\{\mu, \frac{\delta}{2}\}$ small enough, such that

$$\tilde{c} := -\varrho_0(1 + |k|_{L_1, -\varrho_0}) + \sqrt{2\pi} \inf_{\nu \in [-\varrho_0, \varrho_0]} g_{\frac{\delta}{2}}(\nu) > 0.$$

Summarizing, we have shown that

$$\operatorname{Re}\langle (it + \varrho)M(it + \varrho)x|x \rangle_{H_0} \geq \min\{c_{\varrho_0}, \tilde{c}\} |D(it + \varrho)x|_{H_0}^2$$

for each $\varrho > -\varrho_0$. Using now $|x|_{H_0} \leq (1 + |k|_{L_1, -\varrho_0}) |D(it + \varrho)x|_{H_0}$ for $t \in \mathbb{R}, \varrho > -\varrho_0$, we derive

$$\operatorname{Re}\langle zM(z)x|x \rangle_{H_0} \geq \min\{c_{\varrho_0}, \tilde{c}\} \frac{1}{\left(1 + |k|_{L_1, -\varrho_0}\right)^2} |x|_{H_0}^2$$

for each $x \in H_0$ and $z \in \mathbb{C}_{\operatorname{Re} > -\varrho_0} \setminus B[0, \delta]$. Hence, the assertion follows from Proposition 2.2.5. \square

We conclude this subsection by providing two examples for classes of kernels, which satisfy Condition 2.3.8. We start with a class of kernels considered in [Prü09], where the exponential and polynomial stability of hyperbolic integro-differential equations is studied.

Proposition 2.3.12. *Let $k \in W_{1,\operatorname{loc}}^1(\mathbb{R}_{>0}; \mathbb{R}) \setminus \{0\}$ such that $k \geq 0$, $k' \leq 0$, $\int_0^\infty k(s) \, ds < 1$ and $k \in L_{1,-\mu}(\mathbb{R}_{\geq 0}; \mathbb{R})$ for some $\mu > 0$. Then k satisfies Condition 2.3.8.*

Proof. Since k is assumed to be real-valued, it trivially satisfies (c) and (d). Moreover, since $|k|_{L_1} < 1$, it suffices to prove (e) by Lemma 2.3.10 (a). First, we claim that

$$k(0+) > 0$$

where $k(0+) = \infty$ is allowed. The limit exists, since

$$k(t) = - \int_t^1 k'(s) \, ds + k(1) \quad (t \in]0, 1[)$$

and since the right-hand side converges as t tends to zero by monotone convergence, so does the left-hand side. Moreover, $k(0+) \geq k(t)$ for each $t \in \mathbb{R}_{>0}$ and thus, $k(0+) > 0$ since otherwise $k = 0$. For showing (e) we fix $\delta > 0$ and consider the function

$$\begin{aligned} f : \mathbb{R}_{\geq \delta} \times \mathbb{R}_{\geq -\mu} &\rightarrow \mathbb{R} \\ (t, \varrho) &\mapsto t \int_0^\infty \sin(ts) e^{-\varrho s} k(s) \, ds, \end{aligned}$$

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which is continuous by dominated convergence. Note that

$$\begin{aligned}
 t \operatorname{Im} \langle \widehat{k}(\mathrm{i} t + \varrho) x | x \rangle_{H_0} &= -t |x|_{H_0}^2 \operatorname{Im} \widehat{k}(\mathrm{i} t + \varrho) \\
 &= t |x|_{H_0}^2 \frac{1}{\sqrt{2\pi}} \int_0^\infty \sin(ts) e^{-\varrho s} k(s) \, ds \\
 &= \frac{1}{\sqrt{2\pi}} f(t, \varrho) |x|_{H_0}^2
 \end{aligned} \tag{2.16}$$

for $t \geq \delta$, $\varrho \geq -\mu$ and $x \in H_0$. We follow the strategy presented in [Prü09, Section 5] to show that there exists $0 < \varrho_0 \leq \mu$, such that

$$\inf_{t \geq \delta} \inf_{\varrho \in [-\varrho_0, \varrho_0]} f(t, \varrho) > 0 \text{ and } \inf_{t \geq \delta} f(t, \varrho) > 0 \quad (\varrho \geq -\varrho_0).$$

Since $k \in L_{1,-\mu}(\mathbb{R}_{\geq 0})$ there exist sequences $(a_n)_{n \in \mathbb{N}}$ and $(b_n)_{n \in \mathbb{N}}$ in $\mathbb{R}_{>0}$ such that $a_n \rightarrow 0$ and $b_n \rightarrow \infty$ and $e^{\mu b_n} k(b_n) \rightarrow 0$ as well as $a_n e^{\mu a_n} k(a_n) \rightarrow 0$ as $n \rightarrow \infty$. The latter gives

$$\int_\varepsilon^\infty |k'(s)| e^{\mu s} \, ds = \lim_{n \rightarrow \infty} \int_\varepsilon^{b_n} -k'(s) e^{\mu s} \, ds = \mu \int_\varepsilon^\infty k(s) e^{\mu s} \, ds + k(\varepsilon) e^{\mu \varepsilon} < \infty,$$

for each $\varepsilon > 0$ by monotone convergence. Hence, $k' \in L_{1,-\mu}(\mathbb{R}_{\geq \varepsilon})$ for each $\varepsilon > 0$. Let $t \geq \delta$, $\varrho \geq -\mu$. Then we have

$$\begin{aligned}
 &\int_{a_n}^{b_n} (\cos(ts) - 1) (k'(s) - \varrho k(s)) e^{-\varrho s} \, ds \\
 &= t \int_{a_n}^{b_n} \sin(ts) e^{-\varrho s} k(s) \, ds + (\cos(tb_n) - 1) e^{-\varrho b_n} k(b_n) - (\cos(ta_n) - 1) e^{-\varrho a_n} k(a_n)
 \end{aligned}$$

for each $n \in \mathbb{N}$. Since $e^{\mu b_n} k(b_n) \rightarrow 0$, we infer $(\cos(tb_n) - 1) e^{-\varrho b_n} k(b_n) \rightarrow 0$ as $n \rightarrow \infty$. Similarly

$$|\cos(ta_n) - 1| e^{-\varrho a_n} k(a_n) \leq t e^{-\varrho a_n} a_n k(a_n) \rightarrow 0 \quad (n \rightarrow \infty)$$

and thus, monotone and dominated convergence gives

$$\begin{aligned}
 f(t, \varrho) &= t \int_0^\infty \sin(ts) e^{-\varrho s} k(s) \, ds \\
 &= \int_0^\infty (\cos(ts) - 1) (k'(s) - \varrho k(s)) e^{-\varrho s} \, ds \\
 &= \int_0^\infty (\cos(ts) - 1) k'(s) e^{-\varrho s} \, ds + \varrho \int_0^\infty (1 - \cos(ts)) k(s) e^{-\varrho s} \, ds.
 \end{aligned} \tag{2.17}$$

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First, we prove that $\inf_{t \geq \delta} f(t, \varrho) > 0$ for $\varrho \geq -\varrho_0$ for some $\varrho_0 > 0$. By the latter equality we see that $f(t, \varrho) \geq 0$ for $\varrho \geq 0, t \geq \delta$. Indeed, it even holds $f(t, \varrho) > 0$ for $t \geq \delta, \varrho \geq 0$, since otherwise $k' = 0$ which contradicts $k \in L_1(\mathbb{R}_{\geq 0}) \setminus \{0\}$. Moreover, we observe that for $\varrho \geq -\mu, t \geq \delta$

$$f(t, \varrho) \geq \int_{\varepsilon}^{\infty} (\cos(ts) - 1) k'(s) e^{-\varrho s} ds + \varrho \int_0^{\infty} (1 - \cos(ts)) k(s) e^{-\varrho s} ds,$$

for each $\varepsilon > 0$ since $k' \leq 0$. Since $(t \mapsto k'(t)\chi_{[\varepsilon, \infty[}(t)) \in L_1(\mathbb{R})$, the Lemma of Riemann-Lebesgue (Remark A.1) yields

$$\liminf_{t \rightarrow \infty} f(t, \varrho) \geq - \int_{\varepsilon}^{\infty} k'(s) e^{-\varrho s} ds + \varrho \int_0^{\infty} e^{-\varrho s} k(s) ds = k(\varepsilon) e^{-\varrho \varepsilon} + \varrho \int_0^{\varepsilon} k(s) e^{-\varrho s} ds$$

for each $\varepsilon > 0$. Thus, $\liminf_{t \rightarrow \infty} f(t, \varrho) \geq k(0+) > 0$ for each $\varrho \geq -\mu$. Moreover, the same argumentation yields that for each $K \subseteq \mathbb{R}_{>-\mu}$ compact there is $M > \delta$ such that

$$\forall t \geq M, \varrho \in K : f(t, \varrho) \geq \frac{k(0+)}{2}, \quad (2.18)$$

according to Lemma 2.3.10 (c) and Proposition A.5 applied to $t \mapsto e^{-\nu t}(k(t) + \chi_{[\varepsilon, \infty[}(t)k'(t))$ for $\nu := \min K$. Since $f(t, \varrho) > 0$ for each $t \geq \delta, \varrho \geq 0$ and $\liminf_{t \rightarrow \infty} f(t, \varrho) > 0$, we derive, using the continuity of $f(t, \cdot)$, that

$$\inf_{t \geq \delta} f(t, \varrho) > 0 \quad (\varrho \geq 0).$$

Furthermore, for $\varrho < 0$ (2.17) yields

$$f(t, \varrho) \geq f(t, 0) + 2\varrho |k|_{L_1, \varrho} \geq \inf_{s \geq \delta} f(s, 0) + 2\varrho |k|_{L_1, \varrho},$$

and since $\inf_{s \geq \delta} f(s, 0) > 0$, we find $0 < \varrho_0 \leq \mu$ such that

$$\inf_{t \geq \delta} f(t, \varrho) > 0 \quad (\varrho \geq -\varrho_0). \quad (2.19)$$

Moreover, we note that by (2.18) we find $M > \delta$ such that

$$\inf_{\varrho \in [-\varrho_0, \varrho_0]} \inf_{t \geq M} f(t, \varrho) > 0.$$

Since f is continuous and attains positive values on $[\delta, M] \times [-\varrho_0, \varrho_0]$ by (2.19), we get

$$c := \inf_{t \geq \delta} \inf_{\varrho \in [-\varrho_0, \varrho_0]} f(t, \varrho) > 0.$$

Hence, setting $g_{\delta} := \chi_{[-\varrho_0, \varrho_0]} \frac{c}{\sqrt{2\pi}}$, we infer from (2.16) and (2.19)

$$t \operatorname{Im} \langle \widehat{k}(\mathbf{i}t + \varrho)x | x \rangle_{H_0} \geq g_{\delta}(\varrho) |x|_{H_0}^2 \quad (\varrho \geq -\varrho_0, t \geq \delta, x \in H_0).$$

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This yields (e) by Lemma 2.3.10 (b). □

Remark 2.3.13. By following the lines of the proof, the latter proposition can easily be generalized to operator-valued kernels $k \in L_{1,-\mu}(\mathbb{R}_{\geq 0}; L(H_0))$ with $|k|_{L_1} < 1$ by assuming that k satisfies Condition 2.3.8 (c) and (d), k is locally absolutely continuous on $\mathbb{R}_{>0}$ and $k(t)$ and $-k'(t)$ are accretive for almost every $t \in \mathbb{R}_{>0}$.

A second example for kernels covered by our approach are so called kernels of positive type.

Definition. Let $k \in L_{1,\text{loc}}(\mathbb{R}_{>0}; \mathbb{R})$. Then k is said to be of *positive type*, if for all $t > 0$ and $f \in L_{2,\text{loc}}(\mathbb{R}_{>0})$ we have that

$$\operatorname{Re} \int_0^t ((k \widehat{*} f)(s))^* f(s) \, ds \geq 0,$$

where

$$(k \widehat{*} f)(s) := \int_0^s k(r) f(s-r) \, dr \quad (s > 0).$$

Moreover, k is said to be of *strict positive type*, if there exists $\varepsilon > 0$ such that $t \mapsto k(t) - \varepsilon e^{-t}$ is of positive type.

We recall the following characterization for kernels of positive type originally presented in [NS76] (in fact for measures). However, we follow the proof given in [GLS90, p. 494].

Proposition 2.3.14. *Let $k \in \bigcap_{\varrho > 0} L_{1,\varrho}(\mathbb{R}_{\geq 0}; \mathbb{R})$. Then k is of positive type if and only if $\operatorname{Re} \widehat{k}(z) \geq 0$ for each $z \in \mathbb{C}_{\operatorname{Re} > 0}$. Furthermore, if $k \in L_1(\mathbb{R}_{\geq 0}; \mathbb{R})$ then k is of positive type if and only if $\operatorname{Re} \widehat{k}(it) \geq 0$ for each $t \in \mathbb{R}$.*

Proof. Assume that k is of positive type. Let $t > 0, z \in \mathbb{C}_{\operatorname{Re} > 0}$ and define $f(s) := e^{-zs}$ for $s \in \mathbb{R}_{\geq 0}$. Then we have

$$\begin{aligned} 0 &\leq \operatorname{Re} \int_0^t ((k \widehat{*} f)(s))^* f(s) \, ds \\ &= \operatorname{Re} \int_0^t \left(\int_0^s k(r) e^{-z(s-r)} \, dr \right)^* e^{-zs} \, ds \\ &= \operatorname{Re} \int_0^t \int_0^s k(r) e^{-z^*(s-r)} \, dr e^{-zs} \, ds \\ &= \operatorname{Re} \int_0^t k(r) e^{z^* r} \int_r^t e^{-2s \operatorname{Re} z} \, ds \, dr. \end{aligned}$$

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Letting t tend to infinity gives

$$\begin{aligned}
0 &\leq \operatorname{Re} \int_0^\infty k(r) e^{z^* r} \int_r^\infty e^{-2s \operatorname{Re} z} \, ds \, dr \\
&= \frac{1}{2 \operatorname{Re} z} \operatorname{Re} \int_0^\infty k(r) e^{z^* r} e^{-2r \operatorname{Re} z} \, dr \\
&= \frac{1}{2 \operatorname{Re} z} \operatorname{Re} \int_0^\infty k(r) e^{-z r} \, dr \\
&= \frac{\sqrt{2\pi}}{2 \operatorname{Re} z} \operatorname{Re} \widehat{k}(z),
\end{aligned}$$

which shows the first implication. Assume now that $\operatorname{Re} \widehat{k}(z) \geq 0$ for each $z \in \mathbb{C}_{\operatorname{Re} > 0}$. We note that this implies that $k*$ is an accretive operator on $H_\varrho(\mathbb{R})$ for each $\varrho > 0$ by Lemma 1.3.8. Let $t > 0$ and $f \in L_{2,\operatorname{loc}}(\mathbb{R}_{\geq 0})$. Then $g := \chi_{[0,t]} f \in \bigcap_{\varrho \geq 0} H_\varrho(\mathbb{R})$ and hence,

$$\begin{aligned}
0 &\leq \operatorname{Re} \langle k * g | g \rangle_{H_\varrho} \\
&= \operatorname{Re} \int_0^t ((k * g)(s))^* f(s) e^{-2\varrho s} \, ds \\
&= \operatorname{Re} \int_0^t ((k \widehat{*} f)(s))^* f(s) e^{-2\varrho s} \, ds
\end{aligned}$$

for each $\varrho > 0$. Letting $\varrho \rightarrow 0$ we derive the assertion by dominated convergence.

Assume now that $k \in L_1(\mathbb{R}_{\geq 0}; \mathbb{R})$. If k is of positive type, we have $\operatorname{Re} \widehat{k}(it + \varrho) \geq 0$ for each $t \in \mathbb{R}$, $\varrho > 0$ by what we have shown above. Letting ϱ tend to 0, we get $\operatorname{Re} \widehat{k}(it) \geq 0$ for $t \in \mathbb{R}$. If conversely, $\operatorname{Re} \widehat{k}(it) \geq 0$ for each $t \in \mathbb{R}$, we may use Lemma 1.3.8 to derive that $k*$ defines an accretive operator on $L_2(\mathbb{R})$. Hence, for $f \in L_{2,\operatorname{loc}}(\mathbb{R}_{\geq 0})$ and $t \geq 0$ we derive that

$$\begin{aligned}
0 &\leq \operatorname{Re} \langle k * \chi_{[0,t]} f | \chi_{[0,t]} f \rangle_{L_2(\mathbb{R})} \\
&= \operatorname{Re} \int_0^t ((k \widehat{*} f)(s))^* f(s) \, ds,
\end{aligned}$$

which shows that k is of positive type. □

In [CS11] the authors consider kernels $k \in L_1(\mathbb{R}_{\geq 0}; \mathbb{R})$, such that the function $\mathbb{R}_{\geq 0} \ni t \mapsto \int_t^\infty k(s) \, ds$ is a kernel of positive type. For those kernels we have the following result.

Lemma 2.3.15 ([CS08, Proposition 2.4]). *Let $k \in L_1(\mathbb{R}_{\geq 0}; \mathbb{R})$. We set*

$$K(t) := \int_t^\infty k(s) \, ds \quad (t \geq 0).$$

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Then, if K is of positive type, we have that

$$t \operatorname{Im} \widehat{k}(it + \varrho) \leq 0 \quad (t \in \mathbb{R}, \varrho \geq 0).$$

If $K \in L_1(\mathbb{R}_{\geq 0})$, then also the converse implication holds true.

Proof. Assume that K is of positive type. As $K \in L_\infty(\mathbb{R}_{\geq 0}; \mathbb{R})$ we have that $\operatorname{Re} \widehat{K}(z) \geq 0$ for each $z \in \mathbb{C}_{\operatorname{Re} > 0}$ by Proposition 2.3.14. For $z \in \mathbb{C}_{\operatorname{Re} > 0}$ we have that

$$\begin{aligned} \widehat{K}(z) &= \frac{1}{\sqrt{2\pi}} \int_0^\infty K(s) e^{-zs} \, ds \\ &= \frac{1}{\sqrt{2\pi}} \int_0^\infty \int_s^\infty k(r) \, dr e^{-zs} \, ds \\ &= \frac{1}{\sqrt{2\pi}} \frac{1}{z} \left(- \int_0^\infty k(s) e^{-zs} \, ds + \int_0^\infty k(r) \, dr \right) \\ &= \frac{1}{z} \left(\frac{1}{\sqrt{2\pi}} \int_0^\infty k(s) \, ds - \widehat{k}(z) \right), \end{aligned} \tag{2.20}$$

which yields that

$$\begin{aligned} 0 &\leq \operatorname{Re} \widehat{K}(it + \varrho) \\ &= \frac{1}{|it + \varrho|^2} \left(\varrho \frac{1}{\sqrt{2\pi}} \int_0^\infty k(s) \, ds - \varrho \operatorname{Re} \widehat{k}(it + \varrho) - t \operatorname{Im} \widehat{k}(it + \varrho) \right), \end{aligned}$$

for each $t \in \mathbb{R}, \varrho > 0$. Letting ϱ tend to 0, we infer

$$0 \leq -\frac{1}{t} \operatorname{Im} \widehat{k}(it)$$

which yields gives $t \operatorname{Im} \widehat{k}(it) \leq 0$ for each $t \in \mathbb{R}$. The assertion now follows from Lemma 1.3.10 with $d = 0$. Assume now that $K \in L_1(\mathbb{R}_{\geq 0})$ and

$$t \operatorname{Im} \widehat{k}(it + \varrho) \leq 0 \quad (t \in \mathbb{R}, \varrho \geq 0).$$

In particular we get $t \operatorname{Im} \widehat{k}(it) \leq 0$ and hence, using (2.20), we obtain

$$\operatorname{Re} \widehat{K}(it) = -\operatorname{Re} \frac{1}{it} \widehat{k}(it) = -\frac{1}{t} \operatorname{Im} \widehat{k}(it) \geq 0 \quad (t \in \mathbb{R}).$$

Thus, the assertion follows from Proposition 2.3.14. \square

In [CS11] the exponential stability of a class of hyperbolic semilinear integro-differential equations is studied for kernels $k \in L_{1,-\alpha}(\mathbb{R}_{\geq 0}; \mathbb{R})$ for some $\alpha > 0$, satisfying $\int_0^\infty k(t) \, dt < 1$ and

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$t \mapsto \int_t^\infty e^{\alpha s} k(s) \, ds$ is a kernel of strict positive type. We show that these kernels are also covered by our approach. For doing so, we need the following auxiliary results.

Lemma 2.3.16. *Let $k \in L_{1,-\alpha}(\mathbb{R}_{\geq 0}; \mathbb{R})$ for some $\alpha > 0$, such that $K_\alpha(t) := \int_t^\infty e^{\alpha s} k(s) \, ds$ for $t \geq 0$ is a kernel of strict positive type. Then there is $\varepsilon > 0$ such that*

$$t \operatorname{Im} \widehat{k}(it + \varrho) \leq -\varepsilon \frac{t^2}{t^2 + (\varrho + \alpha + 1)^2} \quad (t \in \mathbb{R}, \varrho \geq -\alpha).$$

Proof. Since K_α is of strict positive type, we find $\varepsilon > 0$, such that $g(t) := K_\alpha(t) - \varepsilon e^{-t} = \int_t^\infty k_\alpha(s) - \varepsilon e^{-s} \, ds$ is of positive type, where we set $k_\alpha(s) := e^{\alpha s} k(s)$. Hence, by Lemma 2.3.15 we have

$$t \operatorname{Im} \left(\widehat{k}_\alpha(it + \varrho) - \varepsilon \frac{1}{it + \varrho + 1} \right) \leq 0 \quad (t \in \mathbb{R}, \varrho \geq 0).$$

Moreover, using $\widehat{k}_\alpha(z) = \widehat{k}(z - \alpha)$ for each $z \in \mathbb{C}_{\operatorname{Re} \geq 0}$, we derive

$$t \operatorname{Im} \widehat{k}(it + \varrho - \alpha) \leq -\varepsilon \frac{t^2}{t^2 + (\varrho + 1)^2} \quad (t \in \mathbb{R}, \varrho \geq 0),$$

which gives the assertion. □

Lemma 2.3.17. *Let $k \in L_1(\mathbb{R}_{\geq 0}; \mathbb{R})$ such that $K(t) := \int_t^\infty k(s) \, ds$ for $t \geq 0$ is a kernel of positive type. Then*

$$\operatorname{Re} \left(z^* \widehat{k}(z) \right) \leq \frac{\operatorname{Re} z}{\sqrt{2\pi}} \int_0^\infty k(s) \, ds \quad (z \in \mathbb{C}_{\operatorname{Re} > 0}).$$

If in addition, $\int_0^\infty k(s) \, ds < 1$, then for each $\varrho > 0$ there exists $c > 0$ such that

$$|1 - \sqrt{2\pi} \widehat{k}(z)| \geq c \quad (z \in \mathbb{C}_{\operatorname{Re} \geq \varrho}).$$

Proof. Since

$$\widehat{K}(z) = \frac{1}{z} \left(\frac{1}{\sqrt{2\pi}} \int_0^\infty k(s) \, ds - \widehat{k}(z) \right) \quad (z \in \mathbb{C}_{\operatorname{Re} > 0}),$$

we derive from Proposition 2.3.14

$$\begin{aligned} 0 &\leq \operatorname{Re} \frac{1}{z} \left(\frac{1}{\sqrt{2\pi}} \int_0^\infty k(s) \, ds - \widehat{k}(z) \right) \\ &= \frac{1}{|z|^2} \left(\frac{\operatorname{Re} z}{\sqrt{2\pi}} \int_0^\infty k(s) \, ds - \operatorname{Re} \left(z^* \widehat{k}(z) \right) \right) \end{aligned}$$

for $z \in \mathbb{C}_{\operatorname{Re} > 0}$, which shows the first assertion. Assume now that $\int_0^\infty k(s) \, ds < 1$ and let $\varrho > 0$. We claim that $\sqrt{2\pi} \widehat{k}(z) \neq 1$ for each $z \in \mathbb{C}_{\operatorname{Re} > 0}$. Indeed, if $\sqrt{2\pi} \widehat{k}(z) = 1$ for some

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$z \in \mathbb{C}_{\text{Re}>0}$, we would have

$$\frac{\text{Re } z}{\sqrt{2\pi}} = \text{Re} \left(z^* \widehat{k}(z) \right) \leq \frac{\text{Re } z}{\sqrt{2\pi}} \int_0^\infty k(s) \, ds,$$

according to what we have shown above. This however contradicts $\int_0^\infty k(s) \, ds < 1$ and thus, $\sqrt{2\pi} \widehat{k}(z) \neq 1$ for $z \in \mathbb{C}_{\text{Re}>0}$. Since $|k|_{L_{1,\nu}} \rightarrow 0$ as $\nu \rightarrow \infty$, we find $\nu > 0$ such that

$$|1 - \sqrt{2\pi} \widehat{k}(z)| \geq 1 - |k|_{L_{1,\nu}} \geq \frac{1}{2} \quad (z \in \mathbb{C}_{\text{Re}>\nu}).$$

Moreover, by Proposition A.5 there exists some $M > 0$ such that

$$|1 - \sqrt{2\pi} \widehat{k}(it + \mu)| \geq \frac{1}{2} \quad (|t| > M, \varrho \leq \mu \leq \nu).$$

By the continuity of \widehat{k} and the fact that $\sqrt{2\pi} \widehat{k}(z) \neq 1$ for each $z \in \mathbb{C}_{\text{Re}>0}$, we infer that there is some $\tilde{c} > 0$ such that

$$|1 - \sqrt{2\pi} \widehat{k}(it + \mu)| \geq \tilde{c} \quad (|t| \leq M, \varrho \leq \mu \leq \nu).$$

Thus, the second assertion follows by setting $c := \min\{\tilde{c}, \frac{1}{2}\}$. □

We can now prove that the kernels considered in [CS11] satisfy Condition 2.3.8.

Proposition 2.3.18. *Let $k \in L_{1,-\alpha}(\mathbb{R}_{\geq 0}; \mathbb{R})$ such that $K_\alpha(t) := \int_t^\infty e^{\alpha s} k(s) \, ds$ for $t \geq 0$ defines a kernel of strict positive type. Moreover, assume that $\int_0^\infty k(s) \, ds < 1$. Then k satisfies Condition 2.3.8.*

Proof. Obviously, k satisfies (c) and (d) since it is assumed to be real-valued. By Lemma 2.3.16 there is some $\varepsilon > 0$ such that

$$t \, \text{Im} \widehat{k}(it + \varrho) \leq -\varepsilon \frac{t^2}{t^2 + (\varrho + \alpha + 1)^2} \quad (t \in \mathbb{R}, \varrho \geq -\alpha).$$

Hence, setting $g_\delta(\varrho) := \varepsilon \frac{\delta^2}{\delta^2 + (\varrho + \alpha + 1)^2}$ for $\varrho \geq -\alpha$ and $\delta > 0$, we infer

$$t \, \text{Im} \langle \widehat{k}(it + \varrho)x | x \rangle_{H_0} = -t \, \text{Im} \widehat{k}(it + \varrho) |x|_{H_0}^2 \geq g_\delta(\varrho) |x|_{H_0}^2 \quad (x \in H_0, |t| \geq \delta, \varrho \geq -\alpha),$$

which is (e). Thus, it remains to show (a) and (b). For doing so, we show that for each $\mu > -\alpha$ the kernel $K_\mu(t) := \int_t^\infty e^{-\mu s} k(s) \, ds$ is of positive type. First we note that

$$t \, \text{Im} \widehat{k}(it + \varrho) \leq 0 \quad (\varrho \geq -\mu).$$

Moreover, $K_\mu \in L_1(\mathbb{R}_{\geq 0})$ since $k \in L_{1,-\alpha}(\mathbb{R}_{\geq 0})$ and hence, K_μ is of positive type according to Lemma 2.3.15. Using the continuity of

$$\varepsilon \mapsto \int_0^\infty e^{\varepsilon t} k(t) \, dt,$$

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we find some $0 < \varepsilon < \alpha$ such that $\int_0^\infty e^{\varepsilon t} k(t) dt < 1$. Let $0 < \mu < \varepsilon$. Since K_ε is of positive type, we can apply Lemma 2.3.17 to find $c > 0$ such that

$$|1 - \sqrt{2\pi}\widehat{k}(z)| \geq c \quad (z \in \mathbb{C}_{\operatorname{Re} \geq -\mu}),$$

which implies (a). Finally, using that K_0 is of positive type, we obtain by Lemma 2.3.17

$$\operatorname{Re} \left(z^* \widehat{k}(z) \right) \leq \frac{\operatorname{Re} z}{\sqrt{2\pi}} \int_0^\infty k(s) ds \quad (z \in \mathbb{C}_{\operatorname{Re} > 0}).$$

Thus, for $\varrho_0 > 0$ we have that

$$\begin{aligned} \operatorname{Re} \langle (it + \varrho)(1 - \sqrt{2\pi}\widehat{k}(it + \varrho)^*)x | x \rangle_{H_0} &= \left(\varrho - \sqrt{2\pi} \operatorname{Re} \left((-it + \varrho)\widehat{k}(it + \varrho) \right) \right) |x|_{H_0}^2 \\ &\geq \left(\varrho - \varrho \int_0^\infty k(s) ds \right) |x|_{H_0}^2 \\ &\geq \varrho_0 \left(1 - \int_0^\infty k(s) ds \right) |x|_{H_0}^2, \end{aligned}$$

for each $x \in H_0, t \in \mathbb{R}, \varrho \geq \varrho_0$, which shows (b). \square

2.4. Notes

The main idea for the exponential stability of evolutionary problems is to study the Fourier-Laplace transformed solution operator, which is an \mathcal{H}^∞ -function on some right half plane and to look for analytic and bounded extensions on a bigger right half plane containing the imaginary axis. This idea is not new and was broadly applied, especially in the framework of C_0 -semigroups. We just mention the famous Gearhart-Prüß Theorem [Prü84], which states that for a C_0 -semigroup on a Hilbert space the growth bound and the abscissa of boundedness coincide. The main reason, why this approach works is that we are dealing with Hilbert space valued functions, since only in this case the Fourier transform (or the Fourier-Laplace transform) becomes unitary, see [Kwa72]. Indeed, the Gearhart-Prüß Theorem is false for general Banach spaces, see for instance the example in [GVW81], where a semigroup with growth bound 0 and abscissa of boundedness $-\infty$ is given. However, under additional assumptions on the semigroup and/or the underlying Banach space, one can show the equality of growth and abscissa of boundedness. Examples are: eventually norm continuous semigroups [EN00, Chapter V, Theorem 1.10], positive semigroups on ordered Banach spaces with normal cone [Neu86] (see also [ABHN11, Theorem 5.3.1]) and positive semigroups on $L_p(\Omega)$ for a σ -finite measure space Ω [Wei95] (see also [ABHN11, Theorem 5.3.6]). We emphasize once again that our notion of exponential stability (which means that the abscissa of boundedness is negative) does not yield an exponential decay for the solutions of evolutionary problems unless the right-hand side is more regular. So, the main point in the cited theorems is that for suitable Cauchy problems, where one can associate a C_0 -semigroup, our notion of exponential stability indeed yields the exponential decay of the solution. We will address the question whether

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a similar result holds for a class of evolutionary problems in the next chapter. It should be noted that our notion of exponential stability defined by the invariance of suitable spaces has a counterpart in the theory of C_0 -semigroups, namely Datko's Theorem ([Dat72], see also [ABHN11, Theorem 5.1.2]). This theorem states that a C_0 -semigroup with generator A on a Banach space X is exponentially stable if and only if all solutions u of the corresponding inhomogeneous Cauchy-Problem

$$\partial_t u = Au + f, \quad u(0) = 0,$$

belong to $L_p(\mathbb{R}_{\geq 0}; X)$ providing that $f \in L_p(\mathbb{R}_{\geq 0}; X)$ for some (or equivalently all) $p \in [1, \infty[$. In other words, the associated solution operator leaves the space $L_p(\mathbb{R}_{\geq 0}; X)$ invariant.

We remark that exponential stability is the perhaps strongest and simplest notion of stability for evolutionary problems. It would be interesting to study other notions of stability like polynomial decay of solutions or simply their convergence to 0. In the framework of C_0 -semigroups, results in these directions are already known. We mention the famous Arendt-Batty-Lyubich-Vũ Theorem (see [AB88] and [LV88]) for the strong stability of semigroups, i.e. $T(t) \rightarrow 0$ strongly as $t \rightarrow \infty$. For the polynomial stability of semigroups on Hilbert spaces we refer to [BT10], where the polynomial stability is characterized in terms of resolvent estimates for the generator. A similar result linking resolvent estimates of the generator and the asymptotic behaviour of the semigroup in a Banach space setting is the Theorem of Batty-Duyckarts, [BD08]. The question, which arises naturally is whether those or similar results can be carried over to evolutionary problems. This is not known so far and worthy to be studied in future.

3. Initial conditions for evolutionary problems

In this last chapter we address the question of how to formulate initial value problems in the framework of evolutionary problems. As evolutionary problems are operator equations, which should hold on the whole real line as “time horizon”, we need to inspect how initial conditions could be imposed in this setting. It turns out that particular distributional right-hand sides and the causality of the solution operators can be used to formulate initial conditions (see e.g. [PM11]). The distributions, which are needed to formulate initial value problems, will be introduced in the first section of this chapter. Furthermore, in the theory of delay equations, it is common to impose histories instead of initial values (see e.g. [Hal71, Web76, BP05]). Since problems with memories are also covered by evolutionary problems, one needs to inspect how histories can be formulated within the present framework and what is a suitable class of histories. We answer these questions in the second section of this chapter. The third section is devoted to the regularity of initial value problems. In particular, we derive conditions on the material law M and the operator A that allows us to associate a strongly continuous semigroup to the problem under consideration. For the theory of strongly continuous semigroups we refer to the monographs [EN00, ABHN11]. Moreover, we show that for those evolutionary problems, where a semigroup can be associated with, the exponential stability presented in Chapter 2 yields the classical exponential stability, as it is introduced in the theory of semigroups under suitable assumptions on the material law. In the last section we discuss several examples and study their regularity and exponential stability.

3.1. Extrapolation spaces

Throughout this section let H be a Hilbert space. In this section we recall the concept of extrapolation spaces, define the space $H_q^{-1}(\mathbb{R}; H)$ and discuss some of its properties. We begin with the definition of extrapolation spaces associated with an operator C .

Definition. Let $C : D(C) \subseteq H \rightarrow H$ be a densely defined closed linear operator. We assume that C is boundedly invertible. Then we define

$$H^1(C) := (D(C), |C \cdot|_H),$$

which, due to the closedness of C , is again a Hilbert space with inner product

$$\langle x|y \rangle_{H^1(C)} := \langle Cx|Cy \rangle_H.$$

Moreover, we define

$$H^{-1}(C) := (H, \widetilde{|C^{-1} \cdot|_H}),$$

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i.e. the completion of H with respect to the norm $x \mapsto |C^{-1}x|_H$. We call $(H^1(C), H, H^{-1}(C))$ the *Gelfand triple associated with C* .

Remark 3.1.1. Note that by definition $C : H^1(C) \rightarrow H$ is an isometry and onto by assumption, hence a unitary operator. Moreover

$$C : D(C) \subseteq H \rightarrow H^{-1}(C)$$

is isometric as well and has a dense range by assumption. Thus, there is a unique extension of C , which we will again denote by C , such that

$$C : H \rightarrow H^{-1}(C)$$

is unitary. We note that we can continue the process of defining extrapolation spaces by using powers of C . This results in so-called Sobolev chains, or more general, Sobolev lattices, see [PM11, Chapter 2].

The following lemma provides another way of defining the space $H^{-1}(C)$.

Lemma 3.1.2. *Let $C : D(C) \subseteq H \rightarrow H$ densely defined closed linear and boundedly invertible. Then so is C^* and*

$$\begin{aligned} U : H^{-1}(C) &\rightarrow H^1(C^*)' \\ x &\mapsto (y \mapsto \langle C^{-1}x | C^*y \rangle_H) \end{aligned}$$

is unitary, if we equip $H^1(C^*)$ with the linear structure

$$(\lambda\varphi + \psi)(y) := \lambda^*\varphi(y) + \psi(y) \quad (\lambda \in \mathbb{C}, \varphi, \psi \in H^1(C^*)', y \in H^1(C^*)).$$

Proof. It is clear that C^* is densely defined closed and linear. Moreover, $(C^*)^{-1} = (C^{-1})^*$, which yields that C^* is boundedly invertible. Let now $w, x \in H^{-1}(C)$ and $\lambda \in \mathbb{C}$. Then

$$\begin{aligned} U(\lambda x + w)(y) &= \langle C^{-1}(\lambda x + w) | C^*y \rangle_H \\ &= \lambda^*U(x)(y) + U(w)(y) \\ &= (\lambda U(x) + U(w))(y), \end{aligned}$$

for each $y \in H^1(C^*)$, which shows the linearity of U . Moreover, we estimate

$$\|U(x)\| = \sup_{|y|_{H^1(C^*)}=1} |\langle C^{-1}x | C^*y \rangle_H| \leq |C^{-1}x|_H = |x|_{H^{-1}(C)}$$

and

$$|x|_{H^{-1}(C)}^2 = \langle C^{-1}x | C^{-1}x \rangle_H = U(x)((C^*)^{-1}C^{-1}x) \leq \|U(x)\| |C^{-1}x|_H = \|U(x)\| |x|_{H^{-1}(C)},$$

which gives the isometry of U . For showing that U is onto, let $\varphi \in H^1(C^*)'$. By the Riesz representation theorem there is $u \in H^1(C^*)$ such that

$$\varphi(y) = \langle u | y \rangle_{H^1(C^*)} = \langle C^*u | C^*y \rangle_H \quad (y \in H^1(C^*)).$$

Hence, setting $x := CC^*u \in H^{-1}(C)$, we obtain $U(x) = \varphi$. □

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Remark 3.1.3. The latter lemma also gives that the inner product in H has a natural extension to a continuous sesquilinearform on $H^{-1}(C) \times H^1(C^*)$.

Further on, we do not distinguish between the spaces $H^{-1}(C)$ and $H^1(C^*)'$.

Lemma 3.1.4. *Let $C : D(C) \subseteq H \rightarrow H$ densely defined closed linear and let $\lambda, \mu \in \varrho(C)$. Then*

$$\begin{aligned} H^1(C - \lambda) &\cong H^1(C - \mu) \\ H^{-1}(C - \lambda) &\cong H^{-1}(C - \mu). \end{aligned}$$

Proof. Obviously $H^1(C - \lambda) = D(C) = H^1(C - \mu)$ as a set. We need show that $\text{id} : H^1(C - \lambda) \rightarrow H^1(C - \mu)$ is continuous. The latter however is clear, since for $x \in D(C)$ we have that

$$|x|_{H^1(C-\mu)} = |(C - \mu)x|_H \leq |(C - \lambda)x|_H + |\lambda - \mu||x|_H \leq (1 + |\lambda - \mu| \|(C - \lambda)^{-1}\|) |x|_{H^1(C-\lambda)}.$$

By interchanging λ and μ in the latter computation, we infer $H^1(C - \lambda) \cong H^1(C - \mu)$. By what we have shown so far, we get that $H^1(C^* - \lambda^*) \cong H^1(C^* - \mu^*)$, since $\lambda^*, \mu^* \in \varrho(C^*)$. Thus, $H^1(C^* - \lambda^*)' \cong H^1(C^* - \mu^*)'$ and hence, $H^{-1}(C - \lambda) \cong H^{-1}(C - \mu)$ by Lemma 3.1.2. \square

Example 3.1.5. Let $\varrho \in \mathbb{R} \setminus \{0\}$ and set $C = \partial_{0,\varrho}$, which is a densely defined closed linear boundedly invertible operator on $H_\varrho(\mathbb{R}; H)$. We set

$$\begin{aligned} H_\varrho^1(\mathbb{R}; H) &:= H^1(\partial_{0,\varrho}), \\ H_\varrho^{-1}(\mathbb{R}; H) &:= H^{-1}(\partial_{0,\varrho}). \end{aligned}$$

Moreover, we set

$$\begin{aligned} H_0^1(\mathbb{R}; H) &:= H^1(\partial_{0,0} + 1), \\ H_0^{-1}(\mathbb{R}; H) &:= H^{-1}(\partial_{0,0} + 1). \end{aligned}$$

For $\varrho > 0$ we aim to compute $\partial_{0,\varrho}\chi_{\mathbb{R}_{\geq t}}x \in H_\varrho^{-1}(\mathbb{R}; H)$ for $t \in \mathbb{R}, x \in H$. For doing so, let $y \in C_c^\infty(\mathbb{R}; H) \subseteq H^1(\partial_{0,\varrho}^*)$. Then

$$\begin{aligned} \langle \partial_{0,\varrho}\chi_{\mathbb{R}_{\geq t}}x | y \rangle_{H^{-1}(\partial_{0,\varrho}) \times H^1(\partial_{0,\varrho}^*)} &= \langle \chi_{\mathbb{R}_{\geq t}}x | \partial_{0,\varrho}^*y \rangle_{H_\varrho(\mathbb{R}; H)} \\ &= \int_t^\infty \langle x | (-y'(s) + 2\varrho y(s)) \rangle_H e^{-2\varrho s} \, ds \\ &= \langle x | y(t) \rangle_H e^{-2\varrho t} \end{aligned}$$

and thus, by the density of $C_c^\infty(\mathbb{R}; H)$ in $H^1(\partial_{0,\varrho}^*)$ we get

$$\partial_{0,\varrho}\chi_{\mathbb{R}_{\geq t}}x = e^{-2\varrho t} \delta_t x,$$

where δ_t denotes the Dirac-Delta distribution in t .

Next, we extend the Fourier-Laplace transform to the space $H_\varrho^{-1}(\mathbb{R}; H)$.

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Proposition 3.1.6. *Let $\varrho \in \mathbb{R} \setminus \{0\}$. Then $\text{im} + \varrho : D(\text{im}) \subseteq L_2(\mathbb{R}; H) \rightarrow L_2(\mathbb{R}; H)$ is boundedly invertible and*

$$\mathcal{L}_\varrho : H_\varrho(\mathbb{R}; H) \subseteq H_\varrho^{-1}(\mathbb{R}; H) \rightarrow H^{-1}(\text{im} + \varrho)$$

has a unitary extension. Moreover, the operator

$$\mathcal{F} : L_2(\mathbb{R}; H) \subseteq H_0^{-1}(\mathbb{R}; H) \rightarrow H^{-1}(\text{im} + 1)$$

has a unitary extension.

Proof. Let $\varrho \neq 0$. Since im is skew-selfadjoint, we infer the bounded invertibility of $\text{im} + \varrho$. Moreover, we have that

$$\begin{aligned} |\mathcal{L}_\varrho f|_{H^{-1}(\text{im} + \varrho)} &= |(\text{im} + \varrho)^{-1} \mathcal{L}_\varrho f|_{L_2(\mathbb{R}; H)} \\ &= |\mathcal{L}_\varrho \partial_{0, \varrho}^{-1} f|_{L_2(\mathbb{R}; H)} \\ &= |\partial_{0, \varrho}^{-1} f|_{H_\varrho(\mathbb{R}; H)} \\ &= |f|_{H_\varrho^{-1}(\mathbb{R}; H)} \end{aligned}$$

for each $f \in H_\varrho(\mathbb{R}; H)$. Since $\mathcal{L}_\varrho [H_\varrho(\mathbb{R}; H)] = L_2(\mathbb{R}; H) \subseteq H^{-1}(\text{im} + \varrho)$ is dense, the assertion follows. The proof for \mathcal{F} is completely analogous and is therefore omitted. \square

Proposition 3.1.7. *Let $A : D(A) \subseteq H_0 \rightarrow H_1$ be a densely defined closed and linear operator between two Hilbert spaces H_0 and H_1 . Then*

$$A : D(A) \subseteq H_0 \rightarrow H^{-1}(|A^*| + 1)$$

is bounded and thus, it has a unique continuous extension to H_0 .

Proof. We use the relation (see e.g. [Wei80, p. 197 ff.])

$$A|A|x = |A^*|Ax$$

for all elements $x \in D(A^*A)$. Then we have for $x \in D(A^*A)$

$$\begin{aligned} |Ax|_{H^{-1}(|A^*| + 1)} &= |(|A^*| + 1)^{-1} Ax|_{H_1} \\ &= |A(|A| + 1)^{-1} x|_{H_1} \\ &= ||A|(|A| + 1)^{-1} x|_{H_0} \\ &\leq |x|_{H_0}. \end{aligned}$$

This gives the assertion, since $D(A^*A)$ is dense in H_0 . \square

Lemma 3.1.8. *Let $\varrho > 0, t \in \mathbb{R}$ and $f \in H_\varrho^{-1}(\mathbb{R}; H)$. Then $\text{spt } f \subseteq \mathbb{R}_{\geq t}$ (here we mean the usual support of a distribution) if and only if $\text{spt } \partial_{0, \varrho}^{-1} f \subseteq \mathbb{R}_{\geq t}$.*

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Proof. Assume that $\text{spt } f \subseteq \mathbb{R}_{\geq t}$. By continuous extension we obtain

$$\langle f|g \rangle_{H^{-1}(\partial_{0,\varrho}) \times H^1(\partial_{0,\varrho}^*)} = 0$$

for each $g \in H^1(\partial_{0,\varrho})$ with $\text{spt } g \subseteq \mathbb{R}_{< t}$. Let now $\varphi \in C_c^\infty(\mathbb{R}; H)$ with $\text{spt } \varphi \subseteq \mathbb{R}_{< t}$. Then an easy computation yields that

$$\left((\partial_{0,\varrho}^*)^{-1} \varphi \right) (s) = \int_s^\infty \varphi(r) e^{-2\varrho(r-s)} \, dr \quad (s \in \mathbb{R})$$

and consequently $\text{spt } (\partial_{0,\varrho}^*)^{-1} \varphi \subseteq \mathbb{R}_{< t}$. Hence, we obtain

$$\langle \partial_{0,\varrho}^{-1} f | \varphi \rangle_{H_\varrho(\mathbb{R}; H)} = \langle f | (\partial_{0,\varrho}^*)^{-1} \varphi \rangle_{H^{-1}(\partial_{0,\varrho}) \times H^1(\partial_{0,\varrho}^*)} = 0,$$

which gives $\text{spt } \partial_{0,\varrho}^{-1} f \subseteq \mathbb{R}_{\geq t}$. If on the other hand $\text{spt } \partial_{0,\varrho}^{-1} f \subseteq \mathbb{R}_{\geq t}$, we compute for each $\varphi \in C_c^\infty(\mathbb{R}; H)$ with $\text{spt } \varphi \subseteq \mathbb{R}_{< t}$

$$\langle f | \varphi \rangle_{H^{-1}(\partial_{0,\varrho}) \times H^1(\partial_{0,\varrho}^*)} = \langle \partial_{0,\varrho}^{-1} f | \partial_{0,\varrho}^* \varphi \rangle_{H_\varrho(\mathbb{R}; H)} = 0,$$

since $\text{spt } \partial_{0,\varrho}^* \varphi \subseteq \text{spt } \varphi \subseteq \mathbb{R}_{< t}$. □

Our goal is to extend the solution theory for evolutionary problems to the extrapolation space $H_\varrho^{-1}(\mathbb{R}; H)$. For doing so, we need the following auxiliary results.

Proposition 3.1.9. *Let $\varrho \in \mathbb{R}$ and $F : \{it + \varrho; t \in \mathbb{R}\} \rightarrow L(H)$ be bounded and strongly measurable. Then $F(\partial_{0,\varrho})$ leaves the spaces $H_\varrho^1(\mathbb{R}; H)$ invariant and $F(\partial_{0,\varrho}) \in L(H_\varrho^1(\mathbb{R}; H))$. Moreover,*

$$\partial_{0,\varrho} F(\partial_{0,\varrho}) g = F(\partial_{0,\varrho}) \partial_{0,\varrho} g \quad (g \in H_\varrho^1(\mathbb{R}; H))$$

and the operator

$$F(\partial_{0,\varrho}) : H_\varrho(\mathbb{R}; H) \subseteq H_\varrho^{-1}(\mathbb{R}; H) \rightarrow H_\varrho^{-1}(\mathbb{R}; H)$$

is bounded and thus, it has a unique bounded extension to $H_\varrho^{-1}(\mathbb{R}; H)$. Moreover, we have that

$$(\partial_{0,\varrho} - c)^{-1} F(\partial_{0,\varrho}) g = F(\partial_{0,\varrho}) (\partial_{0,\varrho} - c)^{-1} g$$

for each $g \in H_\varrho^{-1}(\mathbb{R}; H)$ and $c \in \varrho(\partial_{0,\varrho})$.

Proof. Let $g \in H_\varrho^1(\mathbb{R}; H)$ and set $u := F(\partial_{0,\varrho}) g$. We need to show that $u \in H_\varrho^1(\mathbb{R}; H)$, which is equivalent to

$$t \mapsto (it + \varrho) (\mathcal{L}_\varrho u) (t) \in L_2(\mathbb{R}; H).$$

Since

$$(\mathcal{L}_\varrho u) (t) = F(it + \varrho) (\mathcal{L}_\varrho g) (t)$$

for almost every $t \in \mathbb{R}$, we infer that

$$(it + \varrho) (\mathcal{L}_\varrho u) (t) = F(it + \varrho) (it + \varrho) (\mathcal{L}_\varrho g) (t) \tag{3.1}$$

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for almost every $t \in \mathbb{R}$ and thus,

$$\int_{\mathbb{R}} |(\mathbf{i}t + \varrho)(\mathcal{L}_\varrho u)(t)|^2 dt \leq \sup_{s \in \mathbb{R}} \|F(\mathbf{i}s + \varrho)\|^2 \int_{\mathbb{R}} |(\mathbf{i}t + \varrho)\mathcal{L}_\varrho g(t)|^2 dt < \infty,$$

since $g \in H_\varrho^1(\mathbb{R}; H)$. Moreover, applying the inverse Fourier-Laplace transform \mathcal{L}_ϱ^* to both sides of (3.1), we obtain

$$\partial_{0,\varrho} F(\partial_{0,\varrho})g = F(\partial_{0,\varrho})\partial_{0,\varrho}g.$$

Consider now $F(\partial_{0,\varrho}) : H_\varrho(\mathbb{R}; H) \subseteq H_\varrho^{-1}(\mathbb{R}; H) \rightarrow H_\varrho^{-1}(\mathbb{R}; H)$ and let $g \in H_\varrho(\mathbb{R}; H), c \in \varrho(\partial_{0,\varrho})$. Since

$$F(\partial_{0,\varrho})g = F(\partial_{0,\varrho})(\partial_{0,\varrho} - c)(\partial_{0,\varrho} - c)^{-1}g = (\partial_{0,\varrho} - c)F(\partial_{0,\varrho})(\partial_{0,\varrho} - c)^{-1}g,$$

by what we have shown above, we get

$$(\partial_{0,\varrho} - c)^{-1}F(\partial_{0,\varrho})g = F(\partial_{0,\varrho})(\partial_{0,\varrho} - c)^{-1}g. \quad (3.2)$$

Thus, using Lemma 3.1.4 we estimate

$$\begin{aligned} |F(\partial_{0,\varrho})g|_{H_\varrho^{-1}(\mathbb{R}; H)} &\leq C|(\partial_{0,\varrho} - c)^{-1}F(\partial_{0,\varrho})g|_{H_\varrho(\mathbb{R}; H)} \\ &= C|F(\partial_{0,\varrho})(\partial_{0,\varrho} - c)^{-1}g|_{H_\varrho(\mathbb{R}; H)} \\ &\leq C\|F(\partial_{0,\varrho})\| |(\partial_{0,\varrho} - c)^{-1}g|_{H_\varrho(\mathbb{R}; H)} \\ &\leq \tilde{C}\|F(\partial_{0,\varrho})\| |g|_{H_\varrho^{-1}(\mathbb{R}; H)}, \end{aligned}$$

for suitable constants $C, \tilde{C} > 0$, which shows the boundedness of $F(\partial_{0,\varrho})$ on $H_\varrho^{-1}(\mathbb{R}; H)$. The last equality now follows from (3.2) by continuous extension. \square

Remark 3.1.10. The latter proposition particularly applies to solution operators of well-posed evolutionary problems. Indeed, let $A : D(A) \subseteq H \rightarrow H$ densely defined closed and linear and $M : D(M) \subseteq \mathbb{C} \rightarrow L(H)$ a linear material law. Assume that the associated evolutionary problem is well-posed. Then, for $\varrho > s_0(M, A)$ we have that

$$\left(\overline{\partial_{0,\varrho} M(\partial_{0,\varrho}) + A} \right)^{-1} = F(\partial_{0,\varrho}),$$

where $F(\mathbf{i}t + \varrho) := ((\mathbf{i}t + \varrho)M(\mathbf{i}t + \varrho) + A)^{-1}$ for $t \in \mathbb{R}$ and hence, Proposition 3.1.9 applies. Moreover, the translation operator τ_h on $H_\varrho(\mathbb{R}; H)$ for $\varrho \in \mathbb{R}$ is another example, since $\tau_h = F(\partial_{0,\varrho})$ for $F(\mathbf{i}t + \varrho) := e^{(\mathbf{i}t + \varrho)h}$.

For later purposes we need a way to compare elements of $H_\varrho^{-1}(\mathbb{R}; H)$ and $H_\mu^{-1}(\mathbb{R}; H)$ for different values $\mu, \varrho \in \mathbb{R}$. This will be done by the following equivalence relation (see also [PTW14a]).

Definition. Let $f \in H_\varrho^{-1}(\mathbb{R}; H), g \in H_\mu^{-1}(\mathbb{R}; H)$ for some $\varrho, \mu \in \mathbb{R}$. Then we set

$$f \sim g :\Leftrightarrow \exists c < \min\{\varrho, \mu\} : (\partial_{0,\varrho} - c)^{-1}f = (\partial_{0,\mu} - c)^{-1}g.$$

We state the following useful observations.

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Lemma 3.1.11. *Let $\mu, \varrho \in \mathbb{R}$ and $f \in H_{\varrho}^{-1}(\mathbb{R}; H), g \in H_{\mu}^{-1}(\mathbb{R}; H)$.*

(a) *The following statements are equivalent:*

- (i) $f \sim g$,
- (ii) *there exists a sequence $(\varphi_n)_{n \in \mathbb{N}}$ in $C_c^\infty(\mathbb{R}; H)$ such that $\varphi_n \rightarrow f$ in $H_{\varrho}^{-1}(\mathbb{R}; H)$ and $\varphi_n \rightarrow g$ in $H_{\mu}^{-1}(\mathbb{R}; H)$ as $n \rightarrow \infty$,*
- (iii) *For all $c < \min\{\varrho, \mu\}$ we have $(\partial_{0,\varrho} - c)^{-1}f = (\partial_{0,\mu} - c)^{-1}g$.*

(b) *If $f \in H_{\varrho}(\mathbb{R}; H), g \in H_{\mu}(\mathbb{R}; H)$ then $f \sim g$ if and only if $f = g$.*

Proof. (a) (i) \Rightarrow (ii): Assume $f \sim g$, i.e. there is some $c < \min\{\varrho, \mu\}$ such that $(\partial_{0,\varrho} - c)^{-1}f = (\partial_{0,\mu} - c)^{-1}g$. In particular, $(\partial_{0,\varrho} - c)^{-1}f \in H_{\varrho}(\mathbb{R}; H) \cap H_{\mu}(\mathbb{R}; H)$ and thus, there exists a sequence $(\psi_n)_{n \in \mathbb{N}}$ in $C_c^\infty(\mathbb{R}; H)$ such that $\psi_n \rightarrow (\partial_{0,\varrho} - c)^{-1}f$ in $H_{\varrho}(\mathbb{R}; H)$ and $H_{\mu}(\mathbb{R}; H)$ as $n \rightarrow \infty$ according to Lemma 1.1.2. We set $\varphi_n := \psi'_n - c\psi_n \in C_c^\infty(\mathbb{R}; H)$. Then we have, using Lemma 3.1.4,

$$\begin{aligned} |\varphi_n - f|_{H_{\varrho}^{-1}(\mathbb{R}; H)} &\leq C|(\partial_{0,\varrho} - c)^{-1}\varphi_n - (\partial_{0,\varrho} - c)^{-1}f|_{H_{\varrho}(\mathbb{R}; H)} \\ &= C|\psi_n - (\partial_{0,\varrho} - c)^{-1}f|_{H_{\varrho}(\mathbb{R}; H)} \rightarrow 0 \end{aligned}$$

and

$$\begin{aligned} |\varphi_n - g|_{H_{\mu}^{-1}(\mathbb{R}; H)} &\leq C|(\partial_{0,\mu} - c)^{-1}\varphi_n - (\partial_{0,\mu} - c)^{-1}g|_{H_{\mu}(\mathbb{R}; H)} \\ &= C|\psi_n - (\partial_{0,\varrho} - c)^{-1}f|_{H_{\mu}(\mathbb{R}; H)} \rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$.

(ii) \Rightarrow (iii): Let $(\varphi_n)_{n \in \mathbb{N}}$ in $C_c^\infty(\mathbb{R}; H)$ be a sequence as in (ii) and $c < \min\{\varrho, \mu\}$. Note that by Lemma 3.1.4 we have that

$$\begin{aligned} (\partial_{0,\varrho} - c)^{-1}\varphi_n &\rightarrow (\partial_{0,\varrho} - c)^{-1}f \text{ in } H_{\varrho}(\mathbb{R}; H) \\ (\partial_{0,\mu} - c)^{-1}\varphi_n &\rightarrow (\partial_{0,\mu} - c)^{-1}g \text{ in } H_{\mu}(\mathbb{R}; H) \end{aligned}$$

as $n \rightarrow \infty$. Choosing a suitable subsequence, we assume without loss of generality that the convergence holds pointwise almost everywhere. Since the function $T : \mathbb{C}_{\operatorname{Re} > \min\{\varrho, \mu\}} \rightarrow \mathbb{C}$ defined by $T(z) := (z - c)^{-1}$ is analytic and bounded, we get by Theorem 1.2.9 that

$$(\partial_{0,\varrho} - c)^{-1}\varphi_n = (\partial_{0,\mu} - c)^{-1}\varphi_n$$

for each $n \in \mathbb{N}$ and thus, also

$$(\partial_{0,\varrho} - c)^{-1}f = (\partial_{0,\mu} - c)^{-1}g.$$

(iii) \Rightarrow (i): This is trivial.

(b) Let $f \in H_{\varrho}(\mathbb{R}; H)$ and $g \in H_{\mu}(\mathbb{R}; H)$. If $f \sim g$ then there is some $c < \min\{\varrho, \mu\}$ such that $(\partial_{0,\varrho} - c)^{-1}f = (\partial_{0,\mu} - c)^{-1}g$. We prove the following auxiliary result. For $h \in$

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$H_\mu^1(\mathbb{R}; H) \cap H_\varrho^1(\mathbb{R}; H)$ we have $\partial_{0,\mu}h = \partial_{0,\varrho}h$. Indeed, for $\varphi \in C_c^\infty(\mathbb{R}; H)$ we have that

$$\begin{aligned} \int_{\mathbb{R}} \langle \partial_{0,\varrho}h(t) | \varphi(t) \rangle dt &= \langle \partial_{0,\varrho}h | e^{2\varrho m} \varphi \rangle_{H_\varrho(\mathbb{R}; H)} \\ &= \langle h | \partial_{0,\varrho}^* (e^{2\varrho m} \varphi) \rangle_{H_\varrho(\mathbb{R}; H)} \\ &= \langle h | -e^{2\varrho m} \varphi' \rangle_{H_\varrho(\mathbb{R}; H)} \\ &= \int_{\mathbb{R}} \langle h(t) | -\varphi'(t) \rangle dt \end{aligned}$$

and the same computation for μ instead of ϱ yields

$$\int_{\mathbb{R}} \langle \partial_{0,\varrho}h(t) | \varphi(t) \rangle dt = \int_{\mathbb{R}} \langle \partial_{0,\mu}h(t) | \varphi(t) \rangle dt.$$

Since this holds for each $\varphi \in C_c^\infty(\mathbb{R}; H)$, we infer $\partial_{0,\varrho}h = \partial_{0,\mu}h$. Applying this result to $h := (\partial_{0,\varrho} - c)^{-1}f \in H_\varrho^1(\mathbb{R}; H) \cap H_\mu^1(\mathbb{R}; H)$, we obtain

$$f = (\partial_{0,\varrho} - c)h = (\partial_{0,\mu} - c)h = g.$$

If on the other hand $f = g$, then Theorem 1.2.9 applied to $T(z) := (z - c)^{-1}$ for $z \in \mathbb{C}_{\operatorname{Re} > \max\{\varrho, \mu\}}$ and some $c < \max\{\varrho, \mu\}$ yields $(\partial_{0,\varrho} - c)^{-1}f = (\partial_{0,\mu} - c)^{-1}g$, i.e., $f \sim g$. \square

Remark 3.1.12. The latter lemma in particular implies that \sim is an equivalence relation on the set $\bigcup_{\varrho \in \mathbb{R}} H_\varrho^{-1}(\mathbb{R}; H)$. Indeed, the reflexivity and symmetry are obvious and the transitivity can easily be obtained by using the equivalence (i) \Leftrightarrow (iii) in Lemma 3.1.11 (a).

With the relation \sim at hand, we are able to prove the independence on the parameter ϱ of the operator $(\partial_{0,\varrho}M(\partial_{0,\varrho}) + A)^{-1}$, established as a bounded operator on $H_\varrho^{-1}(\mathbb{R}; H)$. As in Section 1.2, we formulate this result for analytic and bounded operator-valued functions.

Proposition 3.1.13. *Let $\varrho_0 \in \mathbb{R}$ and $T : \mathbb{C}_{\operatorname{Re} > \varrho_0} \rightarrow L(H)$ be analytic and bounded. Let $\varrho, \mu \geq \varrho_0$ and $f \in H_\varrho^{-1}(\mathbb{R}; H), g \in H_\mu^{-1}(\mathbb{R}; H)$ with $f \sim g$. Then, $T(\partial_{0,\varrho})f \sim T(\partial_{0,\mu})g$.*

Proof. Let $c < \varrho_0$. Then we have $h := (\partial_{0,\varrho} - c)^{-1}f = (\partial_{0,\mu} - c)^{-1}g \in H_\varrho(\mathbb{R}; H) \cap H_\mu(\mathbb{R}; H)$ according to Lemma 3.1.11 (a). Hence, by Proposition 3.1.9 and Theorem 1.2.9 we have that

$$(\partial_{0,\varrho} - c)^{-1}T(\partial_{0,\varrho})f = T(\partial_{0,\varrho})h = T(\partial_{0,\mu})h = (\partial_{0,\mu} - c)^{-1}T(\partial_{0,\mu})g,$$

which shows the assertion. \square

Classical initial value problems are formulated on $\mathbb{R}_{>0}$ as “time horizon”. Since evolutionary problems use the whole real line, we need to find a proper way to restrict the “time horizon” to the positive reals. The key for doing this is the following lemma.

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Lemma 3.1.14. *Let $\varrho > 0$ and $t \in \mathbb{R}$. Consider the operators*

$$\begin{aligned}\chi_{\mathbb{R}_{\geq t}}(\mathbf{m}) : H_{\varrho}(\mathbb{R}; H) &\rightarrow H_{\varrho}(\mathbb{R}; H) \\ f &\mapsto (s \mapsto \chi_{\mathbb{R}_{\geq t}}(s)f(s))\end{aligned}$$

and

$$\begin{aligned}\chi_{\mathbb{R}_{\leq t}}(\mathbf{m}) : H_{\varrho}(\mathbb{R}; H) &\rightarrow H_{\varrho}(\mathbb{R}; H) \\ f &\mapsto (s \mapsto \chi_{\mathbb{R}_{\leq t}}(s)f(s)).\end{aligned}$$

Then

$$\begin{aligned}\chi_{\mathbb{R}_{\geq t}}(\mathbf{m})f &= \partial_{0,\varrho}\chi_{\mathbb{R}_{\geq t}}(\mathbf{m})\partial_{0,\varrho}^{-1}f - e^{-2\varrho t} \left(\partial_{0,\varrho}^{-1}f \right) (t+)\delta_t, \\ \chi_{\mathbb{R}_{\leq t}}(\mathbf{m})f &= \partial_{0,\varrho}\chi_{\mathbb{R}_{\leq t}}(\mathbf{m})\partial_{0,\varrho}^{-1}f + e^{-2\varrho t} \left(\partial_{0,\varrho}^{-1}f \right) (t-)\delta_t\end{aligned}$$

for each $f \in H_{\varrho}(\mathbb{R}; H)$. Note here, that $\partial_{0,\varrho}^{-1}f$ has a continuous representative by Proposition 1.1.8.

Proof. Let $f \in H_{\varrho}(\mathbb{R}; H)$ and set $F(t) := (\partial_{0,\varrho}^{-1}f)(t) = \int_{-\infty}^t f(s) \, ds$ for $t \in \mathbb{R}$. For $g \in C_c^{\infty}(\mathbb{R}; H)$ we then compute using integration by parts

$$\begin{aligned}&\langle \partial_{0,\varrho}\chi_{\mathbb{R}_{\geq t}}(\mathbf{m})\partial_{0,\varrho}^{-1}f | g \rangle_{H^{-1}(\partial_{0,\varrho}) \times H^1(\partial_{0,\varrho}^*)} \\ &= \langle \chi_{\mathbb{R}_{\geq t}}(\mathbf{m})F | \partial_{0,\varrho}^*g \rangle_{H_{\varrho}(\mathbb{R}; H)} \\ &= \int_t^{\infty} \langle F(s) | -g'(s) + 2\varrho g(s) \rangle_H e^{-2\varrho s} \, ds \\ &= \int_t^{\infty} \langle f(s) | g(s) \rangle_H e^{-2\varrho s} \, ds + e^{-2\varrho t} F(t) | g(t) \\ &= \langle \chi_{\mathbb{R}_{\geq t}}(\mathbf{m})f | g \rangle_{H_{\varrho}(\mathbb{R}; H)} + \langle e^{-2\varrho t} F(t+) \delta_t | g \rangle_{H^{-1}(\partial_{0,\varrho}) \times H^1(\partial_{0,\varrho}^*)}.\end{aligned}$$

The latter gives

$$\chi_{\mathbb{R}_{\geq t}}(\mathbf{m})f = \partial_{0,\varrho}\chi_{\mathbb{R}_{\geq t}}(\mathbf{m})\partial_{0,\varrho}^{-1}f - e^{-2\varrho t} \left(\partial_{0,\varrho}^{-1}f \right) (t+)\delta_t.$$

The equality for $\chi_{\mathbb{R}_{\leq t}}(\mathbf{m})f$ follows by arguing analogously. □

The latter representation for the cut-off operators has the advantage that it can be extended to certain elements in $H_{\varrho}^{-1}(\mathbb{R}; H)$ in a canonical way.

Definition. Let $\varrho > 0, t \in \mathbb{R}$. We define

$$\begin{aligned}P_t : D(P_t) \subseteq H_{\varrho}^{-1}(\mathbb{R}; H) &\rightarrow H_{\varrho}^{-1}(\mathbb{R}; H) \\ f &\mapsto \partial_{0,\varrho}\chi_{\mathbb{R}_{\geq t}}(\mathbf{m})\partial_{0,\varrho}^{-1}f - e^{-2\varrho t} \left(\partial_{0,\varrho}^{-1}f \right) (t+)\delta_t\end{aligned}$$

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with¹

$$D(P_t) := \left\{ f \in H_\varrho^{-1}(\mathbb{R}; H); (\partial_{0,\varrho}^{-1}f)(t+) \text{ exists} \right\},$$

as well as

$$\begin{aligned} Q_t : D(Q_t) \subseteq H_\varrho^{-1}(\mathbb{R}; H) &\rightarrow H_\varrho^{-1}(\mathbb{R}; H) \\ f &\mapsto \partial_{0,\varrho}\chi_{\mathbb{R}_{\leq t}}(\mathfrak{m})\partial_{0,\varrho}^{-1}f + e^{-2\varrho t} \left(\partial_{0,\varrho}^{-1}f \right) (t-) \delta_t \end{aligned}$$

with

$$D(Q_t) := \left\{ f \in H_\varrho^{-1}(\mathbb{R}; H); (\partial_{0,\varrho}^{-1}f)(t-) \text{ exists} \right\}.$$

Remark 3.1.15. By Lemma 3.1.14 we have $P_t f = \chi_{\mathbb{R}_{\geq t}}(\mathfrak{m})f$ and $Q_t f = \chi_{\mathbb{R}_{\leq t}}(\mathfrak{m})f$ for $f \in H_\varrho(\mathbb{R}; H)$.

We collect some useful properties for the so introduced operators P_t and Q_t .

Proposition 3.1.16. *Let $\varrho > 0$ and $s, t \in \mathbb{R}$.*

(a) *We have $\delta_s \in D(P_t) \cap D(Q_t)$ and*

$$P_t \delta_s = \begin{cases} \delta_s & \text{if } s > t, \\ 0 & \text{if } s \leq t, \end{cases} \quad Q_t \delta_s = \begin{cases} 0 & \text{if } s \geq t, \\ \delta_s & \text{if } s < t. \end{cases}$$

(b) *It holds*

$$P_t P_s \subseteq P_{\max\{t,s\}}$$

with equality if and only if $t \leq s$.

(c) *It holds*

$$\tau_s P_t = P_{t-s} \tau_s.$$

(d) *For $f \in D(P_t) \cap D(Q_t)$ we have that*

$$f = P_t f + Q_t f + e^{-2\varrho t} \left(\left(\partial_{0,\varrho}^{-1}f \right) (t+) - \left(\partial_{0,\varrho}^{-1}f \right) (t-) \right) \delta_t.$$

(e) *For $f \in H_\varrho^{-1}(\mathbb{R}; H)$ we have that $\text{spt } f \subseteq \mathbb{R}_{\geq t}$ if and only if $f \in D(Q_t)$ with $Q_t f = 0$. Moreover, $\text{spt } f \subseteq \mathbb{R}_{\leq t}$ if and only if $f \in D(P_t)$ with $P_t f = 0$.*

¹For a function $f \in L_{2,\text{loc}}(\mathbb{R}; H)$ we say that $a := f(t+)$ exists for some $t \in \mathbb{R}$, if

$$\forall \varepsilon > 0 \exists \delta > 0 : |f(s) - a| < \varepsilon \quad (s \in]t, t + \delta[\text{ a.e.}).$$

Similarly, we say that $b := f(t-)$ exists for some $t \in \mathbb{R}$, if

$$\forall \varepsilon > 0 \exists \delta > 0 : |f(s) - b| < \varepsilon \quad (s \in]t - \delta, t[\text{ a.e.}).$$

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Proof. (a) Since $\partial_{0,\varrho}^{-1}\delta_s = e^{2\varrho s} \chi_{\mathbb{R}_{\geq s}}$, we infer $\delta_s \in D(P_t) \cap D(Q_t)$. We compute

$$\begin{aligned} P_t \delta_s &= \partial_{0,\varrho} \chi_{\mathbb{R}_{\geq t}}(m) \partial_{0,\varrho}^{-1} \delta_s - e^{-2\varrho t} \left(\partial_{0,\varrho}^{-1} \delta_s \right) (t+) \delta_t \\ &= \partial_{0,\varrho} \chi_{\mathbb{R}_{\geq t}}(m) e^{2\varrho s} \chi_{\mathbb{R}_{\geq s}} - e^{-2\varrho t} \left(e^{2\varrho s} \chi_{\mathbb{R}_{\geq s}} \right) (t+) \delta_t \\ &= e^{2\varrho s} \partial_{0,\varrho} \chi_{\mathbb{R}_{\geq \max\{t,s\}}} - e^{-2\varrho(t-s)} \chi_{\mathbb{R}_{\geq s}}(t+) \delta_t \\ &= e^{2\varrho(s-\max\{t,s\})} \delta_{\max\{t,s\}} - e^{-2\varrho(t-s)} \chi_{\mathbb{R}_{\geq s}}(t+) \delta_t. \end{aligned}$$

Hence, we obtain

$$P_t \delta_s = \begin{cases} \delta_s & \text{if } s > t, \\ 0 & \text{if } s \leq t. \end{cases}$$

The equality for $Q_t \delta_s$ follows by arguing analogously.

(b) Let $f \in D(P_t P_s)$. We have that

$$\partial_{0,\varrho}^{-1} P_s f = \chi_{\mathbb{R}_{\geq s}}(m) \partial_{0,\varrho}^{-1} f - \left(\partial_{0,\varrho}^{-1} f \right) (s+) \chi_{\mathbb{R}_{\geq s}} \quad (3.3)$$

and since $\left(\partial_{0,\varrho}^{-1} P_s f \right) (t+)$ exists, we infer that $\left(\chi_{\mathbb{R}_{\geq s}}(m) \partial_{0,\varrho}^{-1} f \right) (t+)$ exists, too. Consequently, we compute

$$\begin{aligned} P_t P_s f &= \partial_{0,\varrho} \chi_{\mathbb{R}_{\geq t}}(m) \partial_{0,\varrho}^{-1} P_s f - e^{-2\varrho t} \left(\partial_{0,\varrho}^{-1} P_s f \right) (t+) \delta_t \\ &= \partial_{0,\varrho} \chi_{\mathbb{R}_{\geq \max\{t,s\}}}(m) \partial_{0,\varrho}^{-1} f - \partial_{0,\varrho} \left(\partial_{0,\varrho}^{-1} f \right) (s+) \chi_{\mathbb{R}_{\geq \max\{t,s\}}} - \\ &\quad - e^{-2\varrho t} \chi_{\mathbb{R}_{\geq s}}(t+) \left(\partial_{0,\varrho}^{-1} f \right) (t+) \delta_t + e^{-2\varrho t} \left(\partial_{0,\varrho}^{-1} f \right) (s+) \chi_{\mathbb{R}_{\geq s}}(t+) \delta_t \\ &= \partial_{0,\varrho} \chi_{\mathbb{R}_{\geq \max\{t,s\}}}(m) \partial_{0,\varrho}^{-1} f - e^{-2\varrho \max\{t,s\}} \left(\partial_{0,\varrho}^{-1} f \right) (s+) \delta_{\max\{t,s\}} \\ &\quad - e^{-2\varrho t} \chi_{\mathbb{R}_{\geq s}}(t+) \left(\left(\partial_{0,\varrho}^{-1} f \right) (t+) - \left(\partial_{0,\varrho}^{-1} f \right) (s+) \right) \delta_t. \end{aligned}$$

If $t < s$, we get

$$\begin{aligned} P_t P_s f &= \partial_{0,\varrho} \chi_{\mathbb{R}_{\geq s}}(m) \partial_{0,\varrho}^{-1} f - e^{-2\varrho s} \left(\partial_{0,\varrho}^{-1} f \right) (s+) \delta_s \\ &= P_s f, \end{aligned}$$

while for $t \geq s$ we obtain

$$\begin{aligned} P_t P_s f &= \partial_{0,\varrho} \chi_{\mathbb{R}_{\geq t}}(m) \partial_{0,\varrho}^{-1} f - e^{-2\varrho t} \left(\partial_{0,\varrho}^{-1} f \right) (s+) \delta_t - e^{-2\varrho t} \left(\left(\partial_{0,\varrho}^{-1} f \right) (t+) - \left(\partial_{0,\varrho}^{-1} f \right) (s+) \right) \delta_t \\ &= P_t f. \end{aligned}$$

This shows the asserted inclusion $P_t P_s \subseteq P_{\max\{t,s\}}$. We now prove that equality holds if and only if $t \leq s$. Indeed, if $t \leq s$ and $f \in D(P_s)$ we obtain by (3.3) that

$$\left(\partial_{0,\varrho}^{-1} P_s f \right) (t+) = 0$$

and hence, $f \in D(P_t P_s)$. Assume now that $t > s$. We give an example for $f \in D(P_t)$ such

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that $f \notin D(P_t P_s)$. For doing so, we define

$$g(x) := \chi_{\mathbb{R}_{\geq s}}(x) \sin\left(\frac{1}{x-s}\right).$$

Then clearly $g \in H_\varrho(\mathbb{R})$. We set $f := \partial_{0,\varrho} g \in H_\varrho^{-1}(\mathbb{R})$. Then $f \in D(P_t)$ since $\left(\partial_{0,\varrho}^{-1} f\right)(t+) = \sin\left(\frac{1}{t-s}\right)$. However, $f \notin D(P_s)$ since $\lim_{x \rightarrow s+} \sin\left(\frac{1}{x-s}\right)$ does not exist.

(c) Let $f \in H_\varrho^{-1}(\mathbb{R}; H)$. Then we have

$$\begin{aligned} f \in D(P_t) &\Leftrightarrow \left(\partial_{0,\varrho}^{-1} f\right)(t+) \text{ exists} \\ &\Leftrightarrow \left(\tau_s \partial_{0,\varrho}^{-1} f\right)((t-s)+) \text{ exists} \\ &\Leftrightarrow \tau_s f \in D(P_{t-s}). \end{aligned}$$

Moreover, for $f \in D(P_t)$ we have that

$$\begin{aligned} \tau_s P_t f &= \tau_s \left(\partial_{0,\varrho} \chi_{\mathbb{R}_{\geq t}}(m) \partial_{0,\varrho}^{-1} f - e^{-2\varrho t} \left(\partial_{0,\varrho}^{-1} f\right)(t+) \delta_t \right) \\ &= \partial_{0,\varrho} \tau_s \chi_{\mathbb{R}_{\geq t}}(m) \partial_{0,\varrho}^{-1} f - e^{-2\varrho t} \left(\partial_{0,\varrho}^{-1} f\right)(t+) \tau_s \delta_t. \end{aligned}$$

Using now $\tau_s \chi_{\mathbb{R}_{\geq t}}(m) = \chi_{\mathbb{R}_{\geq t-s}}(m) \tau_s$ as well as

$$\begin{aligned} \tau_s \delta_t &= e^{2\varrho t} \partial_{0,\varrho} \tau_s \chi_{\mathbb{R}_{\geq t}} \\ &= e^{2\varrho t} \partial_{0,\varrho} \chi_{\mathbb{R}_{\geq t-s}} \\ &= e^{2\varrho s} \delta_{t-s}, \end{aligned}$$

we derive that

$$\begin{aligned} \tau_s P_t f &= \partial_{0,\varrho} \chi_{\mathbb{R}_{\geq t-s}}(m) \partial_{0,\varrho}^{-1} \tau_s f - e^{-2\varrho(t-s)} \left(\partial_{0,\varrho}^{-1} \tau_s f\right)((t-s)+) \delta_{t-s} \\ &= P_{t-s} \tau_s f. \end{aligned}$$

(d) Let $f \in D(P_t) \cap D(Q_t)$. Then we have that

$$\begin{aligned} P_t f + Q_t f &= \partial_{0,\varrho} \chi_{\mathbb{R}_{\geq t}}(m) \partial_{0,\varrho}^{-1} f - e^{-2\varrho t} \left(\partial_{0,\varrho}^{-1} f\right)(t+) \delta_t + \partial_{0,\varrho} \chi_{\mathbb{R}_{\leq t}}(m) \partial_{0,\varrho}^{-1} f + e^{-2\varrho t} \left(\partial_{0,\varrho}^{-1} f\right)(t-) \delta_t \\ &= f - e^{-2\varrho t} \left(\left(\partial_{0,\varrho}^{-1} f\right)(t+) - \left(\partial_{0,\varrho}^{-1} f\right)(t-) \right) \delta_t. \end{aligned}$$

(e) Let $f \in H_\varrho^{-1}(\mathbb{R}; H)$. We first assume that $\text{spt } f \subseteq \mathbb{R}_{\geq t}$. Then by Lemma 3.1.8 we have $\text{spt } \partial_{0,\varrho}^{-1} f \subseteq \mathbb{R}_{\geq t}$ and thus

$$\left(\partial_{0,\varrho}^{-1} f\right)(t-) = 0.$$

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Hence, $f \in D(Q_t)$ and

$$Q_t f = \partial_{0,\varrho} \chi_{\mathbb{R}_{\leq t}}(\mathbf{m}) \partial_{0,\varrho}^{-1} f + e^{-2\varrho t} \left(\partial_{0,\varrho}^{-1} f \right) (t-) \delta_t = 0.$$

If on the other hand $f \in D(Q_t)$ with $Q_t f = 0$, we compute for $\varphi \in C_c^\infty(\mathbb{R}; H)$ with $\text{spt } \varphi \subseteq \mathbb{R}_{< t}$

$$\begin{aligned} \langle f | \varphi \rangle_{H^{-1}(\partial_{0,\varrho}) \times H^1(\partial_{0,\varrho}^*)} &= \langle \partial_{0,\varrho}^{-1} f | \partial_{0,\varrho}^* \varphi \rangle_{H_\varrho(\mathbb{R}; H)} \\ &= \langle \chi_{\mathbb{R}_{\leq t}}(\mathbf{m}) \partial_{0,\varrho}^{-1} f | \partial_{0,\varrho}^* \varphi \rangle_{H_\varrho(\mathbb{R}; H)} \\ &= \langle Q_t f | \varphi \rangle_{H^{-1}(\partial_{0,\varrho}) \times H^1(\partial_{0,\varrho})} - e^{-2\varrho t} \left(\partial_{0,\varrho}^{-1} f \right) (t-) \varphi(t) \\ &= 0. \end{aligned}$$

This gives $\text{spt } f \subseteq \mathbb{R}_{\geq t}$.

Assume now that $\text{spt } f \subseteq \mathbb{R}_{\leq t}$. Consider the space

$$V := \{ \chi_{\mathbb{R}_{\geq t}} x ; x \in H \} \subseteq H_\varrho(\mathbb{R}; H).$$

Then V is closed and for $g \in H_\varrho(\mathbb{R}; H)$ we have that

$$\begin{aligned} g \in V^\perp &\Leftrightarrow \forall x \in H : \int_t^\infty \langle g(t) | x \rangle_H e^{-2\varrho t} dt = 0 \\ &\Leftrightarrow \int_t^\infty g(t) e^{-2\varrho t} dt = 0. \end{aligned}$$

Let now $g \in V^\perp$. Then

$$\langle \chi_{\mathbb{R}_{\geq t}}(\mathbf{m}) \partial_{0,\varrho}^{-1} f | g \rangle_{H_\varrho(\mathbb{R}; H)} = \langle f | (\partial_{0,\varrho}^*)^{-1} \chi_{\mathbb{R}_{\geq t}}(\mathbf{m}) g \rangle_{H^{-1}(\partial_{0,\varrho}) \times H^1(\partial_{0,\varrho}^*)}$$

and since

$$\left((\partial_{0,\varrho}^*)^{-1} \chi_{\mathbb{R}_{\geq t}}(\mathbf{m}) g \right) (s) = \int_s^\infty \chi_{\mathbb{R}_{\geq t}}(r) g(r) e^{-2\varrho r} dr e^{2\varrho s} = 0$$

for $s \leq t$, we infer that

$$\langle \chi_{\mathbb{R}_{\geq t}}(\mathbf{m}) \partial_{0,\varrho}^{-1} f | g \rangle_{H_\varrho(\mathbb{R}; H)} = 0.$$

Hence, $\chi_{\mathbb{R}_{\geq t}}(\mathbf{m}) \partial_{0,\varrho}^{-1} f \in V$ and thus, there is $x \in H$ with

$$\chi_{\mathbb{R}_{\geq t}}(\mathbf{m}) \partial_{0,\varrho}^{-1} f = \chi_{\mathbb{R}_{\geq t}} x.$$

The latter gives $f \in D(P_t)$ and

$$P_t f = \partial_{0,\varrho} \chi_{\mathbb{R}_{\geq t}}(\mathbf{m}) \partial_{0,\varrho}^{-1} f - e^{-2\varrho t} \delta_t x = 0.$$

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If $f \in D(P_t)$ and $P_t f = 0$ we compute for $\varphi \in C_c^\infty(\mathbb{R}_{>t}; H)$

$$\begin{aligned} \langle f | \varphi \rangle_{H^{-1}(\partial_{0,\varrho}) \times H^1(\partial_{0,\varrho}^*)} &= \langle \partial_{0,\varrho}^{-1} f | \partial_{0,\varrho}^* \varphi \rangle_{H_\varrho(\mathbb{R}; H)} \\ &= \langle \chi_{\mathbb{R}_{\geq t}}(\mathbf{m}) \partial_{0,\varrho}^{-1} f | \partial_{0,\varrho}^* \varphi \rangle_{H_\varrho(\mathbb{R}; H)} \\ &= \langle P_t f | \varphi \rangle_{H^{-1}(\partial_{0,\varrho}) \times H^1(\partial_{0,\varrho})} + e^{-2\varrho t} \left(\partial_{0,\varrho}^{-1} f \right) (t+) \varphi(t) \\ &= 0, \end{aligned}$$

which completes the proof. \square

3.2. Admissible initial values and histories

It is the aim of this section to provide the “correct” spaces for initial values and histories for a given well-posed evolutionary problem. Throughout, let H a Hilbert space and assume that $M : D(M) \subseteq \mathbb{C} \rightarrow L(H)$ is a linear material law and $A : D(A) \subseteq H \rightarrow H$ is densely defined closed and linear, such that the evolutionary problem associated with M and A is well-posed. We first aim to give the following initial value problem a meaningful interpretation in the framework of evolutionary problems:

$$(\partial_{0,\varrho} M(\partial_{0,\varrho}) + A) u = 0 \text{ on } \mathbb{R}_{>0}, \quad (3.4)$$

$$u|_{\mathbb{R}_{<0}} = g. \quad (3.5)$$

Here, $\varrho > s_0(M, A)$ and $g \in \chi_{\mathbb{R}_{\leq 0}}(\mathbf{m})[H_\varrho^1(\mathbb{R}; H)]$, i.e. $g = \chi_{\mathbb{R}_{\leq 0}}(\mathbf{m})w$ for some $w \in H_\varrho^1(\mathbb{R}; H)$ and we are seeking for a solution $u \in H_\varrho^1(\mathbb{R}; H)$ satisfying (3.4) and (3.5). We start by doing some heuristics and ignoring all domain constraints. This will be used to motivate the definition of the space of admissible histories g . As u is known on $\mathbb{R}_{<0}$ by (3.5), we may decompose

$$u = v + g,$$

where $v := \chi_{\mathbb{R}_{\geq 0}}(\mathbf{m})u \in H_\varrho(\mathbb{R}_{\geq 0}; H)$. Then we have

$$v - \chi_{\mathbb{R}_{\geq 0}} g(0-) \in H_\varrho^1(\mathbb{R}; H). \quad (3.6)$$

We replace (3.4) by

$$P_0 (\partial_{0,\varrho} M(\partial_{0,\varrho}) + A) u = 0,$$

where P_0 is the cut-off operator defined in the previous section. Using the decomposition of u we infer

$$P_0 (\partial_{0,\varrho} M(\partial_{0,\varrho}) + A) v = -P_0 (\partial_{0,\varrho} M(\partial_{0,\varrho}) + A) g = -P_0 \partial_{0,\varrho} M(\partial_{0,\varrho}) g,$$

where we have used $P_0 A = A P_0$, since A just acts in H . The left-hand side in the latter equality can be written as

$$\begin{aligned} P_0 \partial_{0,\varrho} M(\partial_{0,\varrho}) v + A v &= \partial_{0,\varrho} \chi_{\mathbb{R}_{\geq 0}}(\mathbf{m}) M(\partial_{0,\varrho}) v - (M(\partial_{0,\varrho}) v) (0+) \delta_0 + A v \\ &= (\partial_{0,\varrho} M(\partial_{0,\varrho}) + A) v - (M(\partial_{0,\varrho}) v) (0+) \delta_0. \end{aligned}$$

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Thus, we end up with an evolutionary problem for the unknown v of the form

$$(\partial_{0,\varrho}M(\partial_{0,\varrho}) + A)v = (M(\partial_{0,\varrho})v)(0+) \delta_0 - P_0 \partial_{0,\varrho}M(\partial_{0,\varrho})g.$$

We note that the right-hand side also depends on v as the value of $(M(\partial_{0,\varrho})v)(0+)$ is unknown. We replace this term by an arbitrary $x \in H$, which will depend on the history g . Since the evolutionary problem is assumed to be well-posed, we get that

$$v = (\partial_{0,\varrho}M(\partial_{0,\varrho}) + A)^{-1} (x\delta_0) - (\partial_{0,\varrho}M(\partial_{0,\varrho}) + A)^{-1} P_0 \partial_{0,\varrho}M(\partial_{0,\varrho})g.$$

Taking condition (3.6) into account, we get that

$$(\partial_{0,\varrho}M(\partial_{0,\varrho}) + A)^{-1} (x\delta_0 - P_0 \partial_{0,\varrho}M(\partial_{0,\varrho})g) - \chi_{\mathbb{R}_{\geq 0}}g(0-) \in H^1_{\varrho}(\mathbb{R}; H)$$

and this condition will be used for the definition of admissible initial values and histories. In order to provide slightly shorter formulas, we introduce the notation

$$S_{\varrho} := \left(\overline{\partial_{0,\varrho}M(\partial_{0,\varrho}) + A} \right)^{-1}.$$

We restrict ourselves to a slightly smaller class of material laws, which are defined as follows.

Definition. Let $M : D(M) \subseteq \mathbb{C} \rightarrow L(H)$ be a linear material law. We call M *regularizing*, if there is $\varrho_0 \in \mathbb{R}_{>0}$ such that $\mathbb{C}_{\operatorname{Re} \geq \varrho_0} \subseteq D(M)$, $M|_{\mathbb{C}_{\operatorname{Re} \geq \varrho_0}}$ is bounded and

$$\forall \varrho \geq \varrho_0, x \in H : (M(\partial_{0,\varrho})\chi_{\mathbb{R}_{\geq 0}}x)(0+) \text{ exists.}$$

Remark 3.2.1. Note that M is regularizing if and only if $\partial_{0,\varrho}M(\partial_{0,\varrho})\chi_{\mathbb{R}_{\geq 0}}x \in D(P_0)$ for each $x \in H$, $\varrho \geq \varrho_0$ for some $\varrho_0 > 0$.

Lemma 3.2.2. *Let M be a linear material law and set*

$$b(M) := \inf \left\{ \varrho_0 \in \mathbb{R}_{\geq 0} ; \mathbb{C}_{\operatorname{Re} \geq \varrho_0} \subseteq D(M), M|_{\mathbb{C}_{\operatorname{Re} \geq \varrho_0}} \text{ bounded} \right\}.$$

Then M is regularizing if and only if $b(M) < \infty$ and

$$\exists \varrho > b(M) \forall x \in H : (M(\partial_{0,\varrho})\chi_{\mathbb{R}_{\geq 0}}x)(0+) \text{ exists.}$$

Proof. If M is regularizing, then the assertion holds trivially. Assume now that $b(M) < \infty$ and

$$\exists \varrho > b(M) \forall x \in H : (M(\partial_{0,\varrho})\chi_{\mathbb{R}_{\geq 0}}x)(0+) \text{ exists.}$$

We need to prove that

$$\forall \varrho > b(M), x \in H : (M(\partial_{0,\varrho})\chi_{\mathbb{R}_{\geq 0}}x)(0+) \text{ exists.}$$

However, since $M|_{\mathbb{C}_{\operatorname{Re} > b(M)}}$ is analytic and bounded on $\mathbb{C}_{\operatorname{Re} \geq b(M) + \varepsilon}$ for each $\varepsilon > 0$, we infer from Theorem 1.2.9 that $M(\partial_{0,\varrho}) = M(\partial_{0,\nu})$ on $H_{\varrho}(\mathbb{R}; H) \cap H_{\nu}(\mathbb{R}; H)$ for each $\varrho, \nu > b(M)$. Since $\chi_{\mathbb{R}_{\geq 0}}x \in \bigcap_{\varrho > 0} H_{\varrho}(\mathbb{R}; H)$, the assertion follows. \square

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Next, we show that for a regularizing material law M we have that for large enough ϱ

$$\partial_{0,\varrho}M(\partial_{0,\varrho})g \in D(P_0)$$

for each $g \in \chi_{\mathbb{R}_{\leq 0}}(\mathfrak{m})[H_\varrho^1(\mathbb{R}; H)]$. For doing so, we need the following small observation.

Lemma 3.2.3. *Let $\varrho > 0$ and $g \in H_\varrho(\mathbb{R}; H)$ with $\text{spt } g \subseteq \mathbb{R}_{\leq 0}$. Then $g \in \chi_{\mathbb{R}_{\leq 0}}(\mathfrak{m})[H_\varrho^1(\mathbb{R}; H)]$ if and only if $g(0-)$ exists and $g + \chi_{\mathbb{R}_{\geq 0}}g(0-) \in H_\varrho^1(\mathbb{R}; H)$.*

Proof. Assume first that $g \in \chi_{\mathbb{R}_{\leq 0}}(\mathfrak{m})[H_\varrho^1(\mathbb{R}; H)]$. Then there exists $w \in H_\varrho^1(\mathbb{R}; H)$ such that $g = \chi_{\mathbb{R}_{\leq 0}}(\mathfrak{m})w$. Since w is continuous by the Sobolev embedding theorem (see Proposition 1.1.8), we infer that $g(0-) = w(0)$ exists. Moreover, we have that

$$\begin{aligned} \partial_{0,\varrho}(g + \chi_{\mathbb{R}_{\geq 0}}g(0-)) &= \partial_{0,\varrho}\chi_{\mathbb{R}_{\leq 0}}(\mathfrak{m})w + \delta_0g(0-) \\ &= \partial_{0,\varrho}\chi_{\mathbb{R}_{\leq 0}}(\mathfrak{m})\partial_{0,\varrho}^{-1}\partial_{0,\varrho}w + \delta_0w(0) \\ &= Q_0\partial_{0,\varrho}w \\ &= \chi_{\mathbb{R}_{\leq 0}}(\mathfrak{m})w \in H_\varrho(\mathbb{R}; H) \end{aligned}$$

and thus, $g + \chi_{\mathbb{R}_{\geq 0}}g(0-) \in H_\varrho^1(\mathbb{R}; H)$.

The reverse implication holds trivially, since $\chi_{\mathbb{R}_{\leq 0}}(\mathfrak{m})(g + \chi_{\mathbb{R}_{\geq 0}}g(0-)) = g$. \square

Lemma 3.2.4. *Let M be a regularizing linear material law and $\varrho > b(M)$. Then for each $g \in \chi_{\mathbb{R}_{\leq 0}}(\mathfrak{m})[H_\varrho^1(\mathbb{R}; H)]$ we have $\partial_{0,\varrho}M(\partial_{0,\varrho})g \in D(P_0)$.*

Proof. Let $g \in \chi_{\mathbb{R}_{\leq 0}}(\mathfrak{m})[H_\varrho^1(\mathbb{R}; H)]$. We need to show that $(M(\partial_{0,\varrho})g)(0+)$ exists. By Lemma 3.2.3 we have that $g(0-)$ exists and $g + \chi_{\mathbb{R}_{\geq 0}}g(0-) \in H_\varrho^1(\mathbb{R}; H)$. Thus, we have that

$$M(\partial_{0,\varrho})g = M(\partial_{0,\varrho})(g + \chi_{\mathbb{R}_{\geq 0}}g(0-)) - M(\partial_{0,\varrho})\chi_{\mathbb{R}_{\geq 0}}g(0-)$$

and since the first summand on the right-hand side is in $H_\varrho^1(\mathbb{R}; H)$ and therefore continuous, and M is regularizing, we infer that $(M(\partial_{0,\varrho})g)(0+)$ exists. \square

With these results at hand, we are now able to define the history space. Throughout, we assume that M is regularizing and we set

$$\begin{aligned} K_\varrho : \chi_{\mathbb{R}_{\leq 0}}(\mathfrak{m})[H_\varrho^1(\mathbb{R}; H)] &\rightarrow H_\varrho^{-1}(\mathbb{R}; H) \\ g &\mapsto P_0\partial_{0,\varrho}M(\partial_{0,\varrho})g. \end{aligned}$$

Definition. Let $\varrho > \max\{s_0(M, A), b(M)\}$. We define

$$\text{His}_\varrho(M, A) := \{g \in \chi_{\mathbb{R}_{\leq 0}}(\mathfrak{m})[H_\varrho^1(\mathbb{R}; H)] ; \exists x \in H : S_\varrho(\delta_0x - K_\varrho g) - \chi_{\mathbb{R}_{\geq 0}}g(0-) \in H_\varrho^1(\mathbb{R}; H)\}$$

the space of *admissible histories* for M and A . Moreover, we define

$$\text{IV}_\varrho(M, A) := \{g(0-) ; g \in \text{His}_\varrho(M, A)\},$$

the space of *admissible initial values* for M and A .

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We first show that the element x used in the definition of $\text{His}_\varrho(M, A)$ is uniquely determined.

Lemma 3.2.5. *Let $g \in \text{His}_\varrho(M, A)$ and $x \in H$ such that*

$$S_\varrho(x\delta_0 - K_\varrho g) - \chi_{\mathbb{R}_{\geq 0}}g(0-) \in H_\varrho^1(\mathbb{R}; H).$$

Then

$$\begin{aligned} x &= (M(\partial_{0,\varrho})\chi_{\mathbb{R}_{\geq 0}}g(0-)) (0+) \\ &= (M(\partial_{0,\varrho})g) (0-) - (M(\partial_{0,\varrho})g) (0+). \end{aligned}$$

Proof. Since

$$S_\varrho(x\delta_0 - K_\varrho g) - \chi_{\mathbb{R}_{\geq 0}}g(0-) \in H_\varrho^1(\mathbb{R}; H),$$

we obtain

$$x\delta_0 - K_\varrho g - (\partial_{0,\varrho}M(\partial_{0,\varrho}) + A) \chi_{\mathbb{R}_{\geq 0}}g(0-) \in H_\varrho(\mathbb{R}; H^{-1}(|A^*| + 1)).$$

Since clearly $A\chi_{\mathbb{R}_{\geq 0}}g(0-) \in H_\varrho(\mathbb{R}; H^{-1}(|A^*| + 1))$ we infer

$$x\delta_0 - K_\varrho g - \partial_{0,\varrho}M(\partial_{0,\varrho})\chi_{\mathbb{R}_{\geq 0}}g(0-) \in H_\varrho(\mathbb{R}; H^{-1}(|A^*| + 1)).$$

We note that due to causality $\partial_{0,\varrho}M(\partial_{0,\varrho})\chi_{\mathbb{R}_{\geq 0}}g(0-) \in N(Q_0)$ and hence, since we have that $\partial_{0,\varrho}M(\partial_{0,\varrho})\chi_{\mathbb{R}_{\geq 0}}g(0-) \in D(P_0)$ by assumption, Proposition 3.1.16 (d) yields

$$\begin{aligned} \partial_{0,\varrho}M(\partial_{0,\varrho})\chi_{\mathbb{R}_{\geq 0}}g(0-) &= P_0\partial_{0,\varrho}M(\partial_{0,\varrho})\chi_{\mathbb{R}_{\geq 0}}g(0-) + \delta_0 (M(\partial_{0,\varrho})\chi_{\mathbb{R}_{\geq 0}}g(0-)) (0+) \\ &= K_\varrho\chi_{\mathbb{R}_{\geq 0}}g(0-) + \delta_0 (M(\partial_{0,\varrho})\chi_{\mathbb{R}_{\geq 0}}g(0-)) (0+). \end{aligned}$$

Thus, we have that

$$\begin{aligned} x\delta_0 - K_\varrho g - \partial_{0,\varrho}M(\partial_{0,\varrho})\chi_{\mathbb{R}_{\geq 0}}g(0-) \\ = \delta_0(x - (M(\partial_{0,\varrho})\chi_{\mathbb{R}_{\geq 0}}g(0-)) (0+)) - K_\varrho (g + \chi_{\mathbb{R}_{\geq 0}}g(0-)) \in H_\varrho(\mathbb{R}; H^{-1}(|A^*| + 1)). \end{aligned}$$

Since $g + \chi_{\mathbb{R}_{\geq 0}}g(0-) \in H_\varrho^1(\mathbb{R}; H)$ by Lemma 3.2.3, we have that $\partial_{0,\varrho}M(\partial_{0,\varrho}) (g + \chi_{\mathbb{R}_{\geq 0}}g(0-)) \in H_\varrho(\mathbb{R}; H)$ and thus,

$$P_0K_\varrho (g + \chi_{\mathbb{R}_{\geq 0}}g(0-)) = \chi_{\mathbb{R}_{\geq 0}}(m)K_\varrho (g + \chi_{\mathbb{R}_{\geq 0}}g(0-)) \in H_\varrho(\mathbb{R}; H).$$

This, in turn, implies

$$\delta_0 (x - (M(\partial_{0,\varrho})\chi_{\mathbb{R}_{\geq 0}}g(0-)) (0+)) \in H_\varrho(\mathbb{R}; H^{-1}(|A^*| + 1))$$

and hence,

$$x = (M(\partial_{0,\varrho})\chi_{\mathbb{R}_{\geq 0}}g(0-)) (0+).$$

Moreover, we have that

$$\begin{aligned} (M(\partial_{0,\varrho})\chi_{\mathbb{R}_{\geq 0}}g(0-)) (0+) &= (M(\partial_{0,\varrho}) (g + \chi_{\mathbb{R}_{\geq 0}}g(0-))) (0+) - (M(\partial_{0,\varrho})g) (0+) \\ &= (M(\partial_{0,\varrho}) (g + \chi_{\mathbb{R}_{\geq 0}}g(0-))) (0-) - (M(\partial_{0,\varrho})g) (0+), \end{aligned}$$

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since $M(\partial_{0,\varrho})(g + \chi_{\mathbb{R}_{\geq 0}}g(0-)) \in H_{\varrho}^1(\mathbb{R}; H)$. Hence, by causality of $M(\partial_{0,\varrho})$, we end up with

$$\begin{aligned} x &= (M(\partial_{0,\varrho})\chi_{\mathbb{R}_{\geq 0}}g(0-))(0+) \\ &= (M(\partial_{0,\varrho})g)(0-) - (M(\partial_{0,\varrho})g)(0+). \end{aligned}$$

□

Definition. We define the operator

$$\begin{aligned} \Gamma_{(M,A)}^{\varrho} : \text{His}_{\varrho}(M, A) &\rightarrow H \\ g &\mapsto (M(\partial_{0,\varrho})g)(0-) - (M(\partial_{0,\varrho})g)(0+). \end{aligned}$$

We now provide a rigorous proof for the heuristics done at the beginning of this section.

Proposition 3.2.6. *Let $\varrho > \max\{s_0(M, A), b(M)\}$ and $g \in \text{His}_{\varrho}(M, A)$. We define*

$$v := S_{\varrho} \left(\Gamma_{(M,A)}^{\varrho} g \delta_0 - K_{\varrho} g \right).$$

Then, for $u := v + g$ we have that $u \in H_{\varrho}^1(\mathbb{R}; H)$ satisfies (3.5). Moreover, u satisfies (3.4) in the sense that

$$\text{spt}(\partial_{0,\varrho}M(\partial_{0,\varrho}) + A)u \subseteq \mathbb{R}_{\leq 0}.$$

Proof. We first show that $v \in H_{\varrho}(\mathbb{R}_{\geq 0}; H)$. Indeed, we have that $v - \chi_{\mathbb{R}_{\geq 0}}g(0-) \in H_{\varrho}^1(\mathbb{R}; H) \subseteq H_{\varrho}(\mathbb{R}; H)$, which gives that $v \in H_{\varrho}(\mathbb{R}; H)$. Moreover, we have that

$$\begin{aligned} \partial_{0,\varrho}^{-1}v &= S_{\varrho}\partial_{0,\varrho}^{-1} \left(\Gamma_{(M,A)}^{\varrho} g \delta_0 - K_{\varrho} g \right) \\ &= S_{\varrho}\partial_{0,\varrho}^{-1} \left(\Gamma_{(M,A)}^{\varrho} g \delta_0 - (\partial_{0,\varrho}\chi_{\mathbb{R}_{\geq 0}}(m)M(\partial_{0,\varrho})g - \delta_0(M(\partial_{0,\varrho})g)(0+)) \right) \\ &= S_{\varrho} \left(\Gamma_{(M,A)}^{\varrho} g \chi_{\mathbb{R}_{\geq 0}} + (M(\partial_{0,\varrho})g)(0+)\chi_{\mathbb{R}_{\geq 0}} - \chi_{\mathbb{R}_{\geq 0}}(m)M(\partial_{0,\varrho})g \right) \end{aligned}$$

and thus, due to causality of S_{ϱ} we deduce that $\text{spt} \partial_{0,\varrho}^{-1}v \subseteq \mathbb{R}_{\geq 0}$. This yields $\text{spt} v \subseteq \mathbb{R}_{\geq 0}$. Hence, $u|_{\mathbb{R}_{< 0}} = g$ and since $v(0+) = g(0-)$ we infer $u \in H_{\varrho}^1(\mathbb{R}; H)$. Let now $\varphi \in C_c^{\infty}(\mathbb{R}_{> 0}; D(A^*))$. Then we have that

$$\begin{aligned} &\langle (\partial_{0,\varrho}M(\partial_{0,\varrho}) + A)u | \varphi \rangle_{H_{\varrho}(\mathbb{R}; H^{-1}(|A^*|+1)) \times H_{\varrho}(\mathbb{R}; H^1(|A^*|+1))} \\ &= \langle u | (\partial_{0,\varrho}M(\partial_{0,\varrho}) + A)^* \varphi \rangle_{H_{\varrho}(\mathbb{R}; H)} \\ &= \langle v | (\partial_{0,\varrho}M(\partial_{0,\varrho}) + A)^* \varphi \rangle_{H_{\varrho}(\mathbb{R}; H)} + \langle g | (\partial_{0,\varrho}M(\partial_{0,\varrho}) + A)^* \varphi \rangle_{H_{\varrho}(\mathbb{R}; H)} \\ &= \langle \Gamma_{(M,A)}^{\varrho} g \delta_0 - K_{\varrho} g | \varphi \rangle_{H^{-1}(\partial_{0,\varrho}) \times H^1(\partial_{0,\varrho}^*)} + \langle g | (\partial_{0,\varrho}M(\partial_{0,\varrho}) + A)^* \varphi \rangle_{H_{\varrho}(\mathbb{R}; H)} \\ &= \langle \partial_{0,\varrho}^{-1} \left(\Gamma_{(M,A)}^{\varrho} g \delta_0 - K_{\varrho} g \right) | \partial_{0,\varrho}^* \varphi \rangle_{H_{\varrho}(\mathbb{R}; H)} + \langle g | (\partial_{0,\varrho}M(\partial_{0,\varrho}))^* \varphi \rangle_{H_{\varrho}(\mathbb{R}; H)}, \end{aligned}$$

where we have used $\text{spt} A^* \varphi \subseteq \mathbb{R}_{\geq 0}$ and $\text{spt} g \subseteq \mathbb{R}_{\leq 0}$. Recalling that

$$\begin{aligned} &\partial_{0,\varrho}^{-1} \left(\Gamma_{(M,A)}^{\varrho} g \delta_0 - K_{\varrho} g \right) \\ &= ((M(\partial_{0,\varrho})g)(0-) - (M(\partial_{0,\varrho})g)(0+))\chi_{\mathbb{R}_{\geq 0}} + (M(\partial_{0,\varrho})g)(0+)\chi_{\mathbb{R}_{\geq 0}} - \chi_{\mathbb{R}_{\geq 0}}(m)M(\partial_{0,\varrho})g \\ &= (M(\partial_{0,\varrho})g)(0-)\chi_{\mathbb{R}_{\geq 0}} - \chi_{\mathbb{R}_{\geq 0}}(m)M(\partial_{0,\varrho})g, \end{aligned}$$

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we deduce that

$$\begin{aligned}
& \langle \partial_{0,\varrho}^{-1} \left(\Gamma_{(M,A)}^\varrho g \delta_0 - K_\varrho g \right) | \partial_{0,\varrho}^* \varphi \rangle_{H_\varrho(\mathbb{R};H)} \\
&= \langle (M(\partial_{0,\varrho})g) (0-) \chi_{\mathbb{R}_{\geq 0}} | \partial_{0,\varrho}^* \varphi \rangle_{H_\varrho(\mathbb{R};H)} - \langle \chi_{\mathbb{R}_{\geq 0}}(m) M(\partial_{0,\varrho})g | \partial_{0,\varrho}^* \varphi \rangle_{H_\varrho(\mathbb{R};H)} \\
&= -\langle \chi_{\mathbb{R}_{\geq 0}}(m) M(\partial_{0,\varrho})g | \partial_{0,\varrho}^* \varphi \rangle_{H_\varrho(\mathbb{R};H)} \\
&= -\langle M(\partial_{0,\varrho})g | \partial_{0,\varrho}^* \varphi \rangle_{H_\varrho(\mathbb{R};H)}
\end{aligned}$$

where we have used $\varphi(0) = 0$ and $\text{spt } \partial_{0,\varrho}^* \varphi \subseteq \text{spt } \varphi \subseteq \mathbb{R}_{>0}$. Summarizing, we obtain that

$$\begin{aligned}
& \langle (\partial_{0,\varrho} M(\partial_{0,\varrho}) + A) u | \varphi \rangle_{H_\varrho(\mathbb{R};H^{-1}(|A^*|+1)) \times H_\varrho(\mathbb{R};H^1(|A^*|+1))} \\
&= -\langle M(\partial_{0,\varrho})g | \partial_{0,\varrho}^* \varphi \rangle_{H_\varrho(\mathbb{R};H)} + \langle g | (\partial_{0,\varrho} M(\partial_{0,\varrho}))^* \varphi \rangle_{H_\varrho(\mathbb{R};H)} \\
&= 0,
\end{aligned}$$

which finishes the proof. \square

In the case that the evolutionary problem associated with M and A does not have memory, the space $\text{IV}_\varrho(M, A)$ should not depend on the admissible histories. More precisely, we should have

$$\text{His}_\varrho(M, A) = \{g \in \chi_{\mathbb{R}_{\leq 0}} [H_\varrho^1(\mathbb{R}; H)] ; g(0-) \in \text{IV}_\varrho(M, A)\}$$

and for $x \in \text{IV}_\varrho(M, A)$ any two functions $g, \tilde{g} \in \text{His}_\varrho(M, A)$ with $\tilde{g}(0-) = g(0-) = x$ should yield the same solution of the initial value problem. Before we study this case, we need to define what we mean by “having no memory”.

Definition. An operator $T : D(T) \subseteq H_\varrho(\mathbb{R}; H) \rightarrow H_\varrho(\mathbb{R}; H)$ is said to be *amnesic*, if for all $f \in D(T)$ and $a \in \mathbb{R}$

$$\text{spt } f \subseteq \mathbb{R}_{\leq a} \Rightarrow \text{spt } Tf \subseteq \mathbb{R}_{\leq a}.$$

Apparently, amnesia and causality are two properties which are strongly related.

Lemma 3.2.7. Let $T : D(T) \subseteq H_\varrho(\mathbb{R}; H) \rightarrow H_\varrho(\mathbb{R}; H)$ and define

$$\begin{aligned}
\sigma_{-1} : L_{2,\text{loc}}(\mathbb{R}; H) &\rightarrow L_{2,\text{loc}}(\mathbb{R}; H) \\
f &\mapsto (t \mapsto f(-t)).
\end{aligned}$$

Then T is amnesic, if and only if $\sigma_{-1} T \sigma_{-1} : D(T \sigma_{-1}) \subseteq H_{-\varrho}(\mathbb{R}; H) \rightarrow H_{-\varrho}(\mathbb{R}; H)$ is causal.

Proof. The proof is obvious. \square

With this lemma at hand, we derive the following representation result for amnesic operators.

Proposition 3.2.8. Let $\varrho \in \mathbb{R}$ and $T : H_\varrho(\mathbb{R}; H) \rightarrow H_\varrho(\mathbb{R}; H)$ bounded, translation invariant. Then T is amnesic if and only if there exists $N : \mathbb{C}_{\text{Re}>0} \rightarrow L(H)$ analytic and bounded, such that

$$\widehat{Tf}(z) = N(-z + \varrho) \widehat{f}(z) \quad (3.7)$$

for each $f \in H_\varrho(\mathbb{R}_{\leq 0}; H)$, $z \in \mathbb{C}_{\text{Re}<\varrho}$.

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Proof. Assume that T is amnesic. By Lemma 3.2.7 we know that $\sigma_{-1}T\sigma_{-1} \in L(H_{-\varrho}(\mathbb{R}; H))$ is causal. Moreover, it is clearly translation invariant and hence, so is

$$S := e^{\varrho m} \sigma_{-1} T \sigma_{-1} (e^{\varrho m})^{-1} : L_2(\mathbb{R}; H) \rightarrow L_2(\mathbb{R}; H), \quad (3.8)$$

where $e^{\varrho m} : H_{-\varrho}(\mathbb{R}; H) \rightarrow L_2(\mathbb{R}; H)$ with $(e^{\varrho m} f)(t) = e^{\varrho t} f(t)$. Now, by Theorem 1.2.11 there exists an analytic and bounded function

$$N : \mathbb{C}_{\operatorname{Re} > 0} \rightarrow L(H)$$

such that

$$\widehat{S}g(z) = N(z)\widehat{g}(z)$$

for each $g \in L_2(\mathbb{R}_{\geq 0}; H)$, $z \in \mathbb{C}_{\operatorname{Re} > 0}$. Then we compute

$$\begin{aligned} \widehat{T}f(z) &= \left(\sigma_{-1} (e^{\varrho m})^{-1} \widehat{S e^{\varrho m} \sigma_{-1} f} \right)(z) \\ &= \left((e^{\varrho m})^{-1} \widehat{S e^{\varrho m} \sigma_{-1} f} \right)(-z) \\ &= \left(\widehat{S e^{\varrho m} \sigma_{-1} f} \right)(-z + \varrho) \\ &= N(-z + \varrho) \left(\widehat{e^{\varrho m} \sigma_{-1} f} \right)(-z + \varrho) \\ &= N(-z + \varrho) \widehat{f}(z) \end{aligned}$$

for each $f \in H_{\varrho}(\mathbb{R}_{\leq 0}; H)$ and $z \in \mathbb{C}_{\operatorname{Re} < \varrho}$, which gives the assertion. Assume now that (3.7) holds. We prove that S given by (3.8) is causal, which would yield the assertion. Let $g \in L_2(\mathbb{R}_{\geq 0}; H)$. We then have

$$\begin{aligned} \widehat{S}g(z) &= \left(e^{\varrho m} \sigma_{-1} T \sigma_{-1} (e^{\varrho m})^{-1} g \right)(z) \\ &= \left(T \sigma_{-1} (e^{\varrho m})^{-1} g \right)(-z + \varrho) \\ &= N(z) \left(\sigma_{-1} (e^{\varrho m})^{-1} g \right)(-z + \varrho) \\ &= N(z) \widehat{g}(z) \end{aligned}$$

for each $z \in \mathbb{C}_{\operatorname{Re} > 0}$. Hence, $\widehat{S}g \in \mathcal{H}^2(\mathbb{C}_{\operatorname{Re} > 0}; H)$ and thus the assertion follows by Theorem A.7. \square

Proposition 3.2.9. *Let $\varrho > 0$ and $T : H_{\varrho}(\mathbb{R}; H) \rightarrow H_{\varrho}^{-1}(\mathbb{R}; H)$ bounded and translation invariant. Then the following statements are equivalent:*

- (i) *For all $\varphi \in C_c^\infty(\mathbb{R}_{< 0}; H)$ we have $T\varphi \in N(P_0)$.*
- (ii) *The operator $(\partial_{0,\varrho}^{-1})^* T$ is amnesic².*

²Note that $(\partial_{0,\varrho}^{-1})^* : H_{\varrho}^{-1}(\mathbb{R}; H) \rightarrow H_{\varrho}(\mathbb{R}; H)$ is bounded, since $\partial_{0,\varrho} \left((\partial_{0,\varrho}^{-1})^* \right) = (-\partial_{0,\varrho}^* + 2\varrho) (\partial_{0,\varrho}^*)^{-1}$ is bounded on $H_{\varrho}(\mathbb{R}; H)$ and $\partial_{0,\varrho}$ is normal.

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(iii) For all $g \in \chi_{\mathbb{R}_{\leq 0}}(\mathfrak{m}) [H_{\varrho}^1(\mathbb{R}; H)]$ we have $Tg \in N(P_0)$.

Proof. (i) \Rightarrow (ii): Let $\varphi \in C_c^\infty(\mathbb{R}_{< 0}; H)$. Then by assumption

$$0 = P_0 T\varphi = \partial_{0,\varrho} \chi_{\mathbb{R}_{\geq 0}}(\mathfrak{m}) \partial_{0,\varrho}^{-1} T\varphi - \delta_0 \left(\partial_{0,\varrho}^{-1} T\varphi \right) (0+).$$

Consequently,

$$\chi_{\mathbb{R}_{\geq 0}}(\mathfrak{m}) \partial_{0,\varrho}^{-1} T\varphi = \chi_{\mathbb{R}_{\geq 0}} x,$$

where $x := \left(\partial_{0,\varrho}^{-1} T\varphi \right) (0+)$. The latter gives $\text{spt} \left(\partial_{0,\varrho}^{-1} \right)^* T\varphi \subseteq \mathbb{R}_{\leq 0}$. Indeed, let $\psi \in C_c^\infty(\mathbb{R}_{> 0}; H)$. Then we compute

$$\begin{aligned} \langle \left(\partial_{0,\varrho}^{-1} \right)^* T\varphi | \psi \rangle_{H_{\varrho}(\mathbb{R}; H)} &= \langle \left(\partial_{0,\varrho}^{-1} \right)^* \partial_{0,\varrho} \partial_{0,\varrho}^{-1} T\varphi | \psi \rangle_{H_{\varrho}(\mathbb{R}; H)} \\ &= \langle \left(\partial_{0,\varrho}^{-1} \right)^* \partial_{0,\varrho}^{-1} T\varphi | \partial_{0,\varrho}^* \psi \rangle_{H_{\varrho}(\mathbb{R}; H)} \\ &= \langle \partial_{0,\varrho}^{-1} T\varphi | \partial_{0,\varrho}^{-1} \partial_{0,\varrho}^* \psi \rangle_{H_{\varrho}(\mathbb{R}; H)} \\ &= \langle \partial_{0,\varrho}^{-1} T\varphi | -\psi + 2\varrho \partial_{0,\varrho}^{-1} \psi \rangle_{H_{\varrho}(\mathbb{R}; H)} \end{aligned}$$

Setting now $\tilde{\psi} := \partial_{0,\varrho}^{-1} \psi \in H_{\varrho}^1(\mathbb{R}; H)$ and noting that due to the causality of $\partial_{0,\varrho}^{-1}$ we have $\text{spt} \tilde{\psi} \subseteq \mathbb{R}_{> 0}$, we infer

$$\begin{aligned} \langle \left(\partial_{0,\varrho}^{-1} \right)^* T\varphi | \psi \rangle_{H_{\varrho}(\mathbb{R}; H)} &= \langle \partial_{0,\varrho}^{-1} T\varphi | -\tilde{\psi}' + 2\varrho \tilde{\psi} \rangle_{H_{\varrho}(\mathbb{R}; H)} \\ &= \langle \chi_{\mathbb{R}_{\geq 0}} x | \partial_{0,\varrho}^* \tilde{\psi} \rangle_{H_{\varrho}(\mathbb{R}; H)} \\ &= \tilde{\psi}(0) = 0. \end{aligned}$$

Using now the density of $C_c^\infty(\mathbb{R}_{< 0}; H)$ in $H_{\varrho}(\mathbb{R}_{\leq 0}; H)$ and the boundedness and translation invariance of $\left(\partial_{0,\varrho}^{-1} \right)^* T$, we derive that $\left(\partial_{0,\varrho}^{-1} \right)^* T$ is amnesic.

(ii) \Rightarrow (iii): Let now $g \in \chi_{\mathbb{R}_{\leq 0}}(\mathfrak{m}) [H_{\varrho}^1(\mathbb{R}; H)]$. We show that $\text{spt} Tg \subseteq \mathbb{R}_{\leq 0}$. For doing so, let $\varphi \in C_c^\infty(\mathbb{R}_{> 0}; H)$. Then

$$\begin{aligned} \langle Tg | \varphi \rangle_{H^{-1}(\partial_{0,\varrho}) \times H^1(\partial_{0,\varrho}^*)} &= \langle \partial_{0,\varrho}^{-1} Tg | \partial_{0,\varrho}^* \varphi \rangle_{H_{\varrho}(\mathbb{R}; H)} \\ &= \langle \partial_{0,\varrho}^{-1} Tg | \partial_{0,\varrho}^{-1} \partial_{0,\varrho}^* \varphi' \rangle_{H_{\varrho}(\mathbb{R}; H)} \\ &= \langle \left(\partial_{0,\varrho}^{-1} \right)^* Tg | \varphi' \rangle_{H_{\varrho}(\mathbb{R}; H)} \\ &= 0, \end{aligned}$$

which gives $\text{spt} Tg \subseteq \mathbb{R}_{\leq 0}$. The assertion now follows from Proposition 3.1.16 (e).

(iii) \Rightarrow (i): This is obvious. □

The latter two propositions now provide a characterization of those material laws, where the history space in fact just depends on the value $g(0-)$ and not on the whole history g .

Proposition 3.2.10. *Let $\varrho > \max\{s_0(M, A), b(M)\}$. Then the following statements are equivalent:*

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(i) We have

$$\text{His}_\varrho(M, A) = \{g \in \chi_{\mathbb{R}_{\leq 0}} [H_\varrho^1(\mathbb{R}; H)] ; g(0-) \in \text{IV}_\varrho(M, A)\}$$

and for each $g \in \text{His}_\varrho(M, A)$ with $g(0-) = 0$ we have that

$$S_\varrho(\delta_0 \Gamma_{(M, A)}^\varrho g - K_\varrho g) = 0.$$

(ii) There exist $M_0, M_1 \in L(H)$ such that

$$M(z) = M_0 + z^{-1} M_1 \quad (z \in D(M) \setminus \{0\}).$$

In this case we have $K_\varrho g = 0$ for each $g \in \chi_{\mathbb{R}_{\leq 0}}(\text{m}) [H_\varrho^1(\mathbb{R}; H)]$.

Proof. (i) \Rightarrow (ii): By assumption, we in particular have $C_c^\infty(\mathbb{R}_{< 0}; H) \subseteq \text{His}_\varrho(M, A)$, since $0 \in \text{IV}_\varrho(M, A)$. Consequently,

$$S_\varrho(\delta_0 \Gamma_{(M, A)}^\varrho \varphi - K_\varrho \varphi) = 0$$

for each $\varphi \in C_c^\infty(\mathbb{R}_{< 0}; H)$ by assumption. Thus,

$$\delta_0 \Gamma_{(M, A)}^\varrho \varphi = K_\varrho \varphi$$

and hence, by Proposition 3.1.16 (b) and (a)

$$K_\varrho \varphi = P_0 \partial_{0, \varrho} M(\partial_{0, \varrho}) \varphi = P_0 \delta_0 \Gamma_{(M, A)}^\varrho \varphi = 0$$

for each $\varphi \in C_c^\infty(\mathbb{R}_{< 0}; H)$. Thus, $\left(\partial_{0, \varrho}^{-1}\right)^* \partial_{0, \varrho} M(\partial_{0, \varrho})$ is amnesic by Proposition 3.2.9. Hence, according to Proposition 3.2.8 there exists $N : \mathbb{C}_{\text{Re} > 0} \rightarrow L(H)$ analytic and bounded such that

$$\left(\left(\partial_{0, \varrho}^{-1}\right)^* \widehat{\partial_{0, \varrho} M(\partial_{0, \varrho}) f}\right)(z) = N(-z + \varrho) \widehat{f}(z) \quad (3.9)$$

for each $f \in H_\varrho(\mathbb{R}_{\leq 0}; H)$ and $z \in \mathbb{C}_{\text{Re} < \varrho}$. Considering the analytic function

$$T : \{z \in \mathbb{C} ; b(M) < \text{Re } z < 2\varrho\} \rightarrow L(H) \\ z \mapsto \frac{z}{-z + 2\varrho} M(z)$$

we get that

$$\left(\partial_{0, \varrho}^{-1}\right)^* \partial_{0, \varrho} M(\partial_{0, \varrho}) = T(\partial_{0, \varrho}).$$

Using that T is bounded on $\{z \in \mathbb{C} ; b(M) + \varepsilon < \text{Re } z < 2\varrho - \varepsilon\}$ for each $\varepsilon > 0$, we can use Lemma 1.2.8 to derive that

$$T(\partial_{0, \varrho}) f = T(\partial_{0, \mu}) f$$

for each $f \in H_\varrho(\mathbb{R}_{\leq 0}; H)$ and $b(M) < \mu < \varrho$. Thus, choosing $f(t) := \sqrt{2\pi} \chi_{\mathbb{R}_{\leq 0}}(t) e^{(\varrho+1)t} x$ for arbitrary $x \in H$, we infer from (3.9)

$$\frac{1}{\varrho + 1 - z} N(-z + \varrho) x = \left(T(\widehat{\partial_{0, \varrho}}) f\right)(z) = \left(T(\widehat{\partial_{0, \text{Re } z}}) f\right)(z) = \frac{z}{-z + 2\varrho} \frac{1}{\varrho + 1 - z} M(z) x$$

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for each $z \in \mathbb{C}$ with $b(M) < \operatorname{Re} z < \varrho$. Hence,

$$N(-z + \varrho) = \frac{z}{-z + 2\varrho} M(z) \quad (b(M) < \operatorname{Re} z < \varrho).$$

Thus,

$$F(z) := \begin{cases} (-z + b(M))M(-z + b(M)) & \text{if } \operatorname{Re} z < 0, \\ (z + 2\varrho - b(M))N(z + \varrho - b(M)) & \text{if } \operatorname{Re} z \geq 0, \end{cases}$$

is an entire function. Moreover

$$|F(z)| \leq C|z| + D \quad (z \in \mathbb{C})$$

for some $C, D \geq 0$. Hence, the function $G(z) := \frac{1}{z}(F(z) - F(0))$ for $z \in \mathbb{C} \setminus \{0\}$ is bounded, which yields $G(z) = B$ for some $B \in L(H)$ and all $z \neq 0$ by the Theorem of Liouville. Hence,

$$F(z) = zB + F(0) \quad (z \in \mathbb{C}).$$

For $\operatorname{Re} z < 0$ we thus have

$$(-z + b(M))M(-z + b(M)) = zB + F(0)$$

and thus,

$$M(z) = \frac{b(M) - z}{z}B + \frac{1}{z}F(0) \quad (z \in \mathbb{C}_{\operatorname{Re} > b(M)}).$$

Setting $M_0 := -B$ and $M_1 := b(M)B + F(0)$, the assertion follows from the identity theorem.

(ii) \Rightarrow (i): Let $g \in \chi_{\mathbb{R}_{\leq 0}}(\mathfrak{m}) [H_{\varrho}^1(\mathbb{R}; H)]$. Then $\left((M_0 + \partial_{0,\varrho}^{-1} M_1) g \right) (0+) = \left(\partial_{0,\varrho}^{-1} M_1 g \right) (0+)$, which exists, since $\partial_{0,\varrho}^{-1} M_1 g \in H_{\varrho}^1(\mathbb{R}; H)$. Thus, $\partial_{0,\varrho} M(\partial_{0,\varrho} g) \in D(P_0)$ and we compute

$$\begin{aligned} K_{\varrho} g &= P_0 \partial_{0,\varrho} M(\partial_{0,\varrho} g) \\ &= \partial_{0,\varrho} \chi_{\mathbb{R}_{\geq 0}}(\mathfrak{m}) \left(M_0 + \partial_{0,\varrho}^{-1} M_1 \right) g - \delta_0 \left(M_0 + \partial_{0,\varrho}^{-1} M_1 g \right) (0+) \\ &= \partial_{0,\varrho} \chi_{\mathbb{R}_{\geq 0}}(\mathfrak{m}) \partial_{0,\varrho}^{-1} M_1 g - \delta_0 \left(\partial_{0,\varrho}^{-1} M_1 g \right) (0+) \\ &= P_0 M_1 g \\ &= \chi_{\mathbb{R}_{\geq 0}}(\mathfrak{m}) M_1 g \\ &= 0. \end{aligned}$$

Let now $x \in \operatorname{IV}_{\varrho}(M, A)$. Then there exists $g \in \operatorname{His}_{\varrho}(M, A)$ such that $g(0-) = x$. Let now $h \in \chi_{\mathbb{R}_{\leq 0}}(\mathfrak{m}) [H_{\varrho}^1(\mathbb{R}; H)]$ with $h(0-) = x$. We compute

$$\begin{aligned} S_{\varrho}(\delta_0 \Gamma_{(M,A)}^{\varrho} g - K_{\varrho} g) &= S_{\varrho} \delta_0 \Gamma_{(M,A)}^{\varrho} g \\ &= S_{\varrho} \delta_0 ((M(\partial_{0,\varrho} g)(0-) - (M(\partial_{0,\varrho} g)(0+))). \end{aligned}$$

Since

$$(M(\partial_{0,\varrho} g)(0-) - (M(\partial_{0,\varrho} g)(0+)) = M_0 g(0-) = M_0 x$$

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we infer

$$S_\varrho(\delta_0\Gamma_{(M,A)}^\varrho g - K_\varrho g) = S_\varrho(\delta_0\Gamma_{(M,A)}^\varrho h - K_\varrho h),$$

and thus, $h \in \text{His}_\varrho(M, A)$. Moreover,

$$S_\varrho(\delta_0\Gamma_{(M,A)}^\varrho g - K_\varrho g) = 0,$$

for $g(0-) = 0$, which gives (i). □

The latter proposition shows that material laws of the form $M(\partial_{0,\varrho}) = M_0 + \partial_{0,\varrho}^{-1}M_1$ are precisely those, where it suffices to prescribe an initial value $g(0-)$ and not a whole history g .

3.3. C_0 -semigroups associated with evolutionary problems

This section is devoted to the regularity of initial value problems as they were introduced in the previous section. More precisely, we characterize those evolutionary problems which allow the definition of an associated C_0 -semigroup on a suitable Hilbert space. The key tool will be the Widder-Arendt Theorem (see [Are87] and Theorem C.6 of this thesis). First of all, we recall the definition of a C_0 -semigroup.

Definition. Let X be a Banach space. A C_0 -semigroup on X is a mapping $T : \mathbb{R}_{\geq 0} \rightarrow L(X)$ such that

- (a) $T(0) = 1$ and for all $t, s \in \mathbb{R}_{\geq 0}$ we have that $T(t+s) = T(t)T(s)$.
- (b) For all $x \in X$ we have that $T(\cdot)x$ is continuous on $\mathbb{R}_{\geq 0}$.

We want to associate a C_0 -semigroup with a well-posed evolutionary problem. For this, we need to find the right Hilbert space X , where the semigroup should act on. Throughout, we assume that $M : D(M) \subseteq \mathbb{C} \rightarrow L(H)$ is a regularizing linear material law, $A : D(A) \subseteq H \rightarrow H$ is densely defined closed and linear and the associated evolutionary problem is well-posed. Again, we set

$$S_\varrho := \overline{(\partial_{0,\varrho}M(\partial_{0,\varrho}) + A)^{-1}}$$

for $\varrho > s_0(M, A)$. We begin with the following proposition.

Proposition 3.3.1. *Let $\varrho > \max\{s_0(M, A), b(M)\}$ and $g \in \text{His}_\varrho(M, A)$. Moreover, we set*

$$v := S_\varrho \left(\delta_0\Gamma_{(M,A)}^\varrho g - K_\varrho g \right)$$

and $u := v + g$. For $t > 0$ we set $\tilde{g} := \chi_{\mathbb{R}_{\leq 0}}(\mathfrak{m})\tau_t u$ and $\tilde{v} := \chi_{\mathbb{R}_{\geq 0}}(\mathfrak{m})\tau_t u$. Then we have $\tilde{g} \in \text{His}_\varrho(M, A)$ and

$$\tilde{v} = S_\varrho \left(\delta_0\Gamma_{(M,A)}^\varrho \tilde{g} - K_\varrho \tilde{g} \right).$$

Proof. We first prove that

$$\text{spt}(\partial_{0,\varrho}M(\partial_{0,\varrho}) + A)\tau_t u \subseteq \mathbb{R}_{\leq 0}. \tag{3.10}$$

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For doing so, let $\varphi \in C_c^\infty(\mathbb{R}_{>0}; D(A^*))$. Then we have

$$\begin{aligned}
& \langle (\partial_{0,\varrho} M(\partial_{0,\varrho}) + A) \tau_t u | \varphi \rangle_{H_\varrho(\mathbb{R}; H^{-1}(|A|+1)) \times H_\varrho(\mathbb{R}; H^1(|A|+1))} \\
&= \langle \tau_t u | (\partial_{0,\varrho} M(\partial_{0,\varrho}) + A)^* \varphi \rangle_{H_\varrho(\mathbb{R}; H)} \\
&= \langle u | \tau_{-t} e^{2\varrho t} (\partial_{0,\varrho} M(\partial_{0,\varrho}) + A)^* \varphi \rangle_{H_\varrho(\mathbb{R}; H)} \\
&= e^{2\varrho t} \langle u | (\partial_{0,\varrho} M(\partial_{0,\varrho}) + A)^* \tau_{-t} \varphi \rangle_{H_\varrho(\mathbb{R}; H)} \\
&= e^{2\varrho t} \langle (\partial_{0,\varrho} M(\partial_{0,\varrho}) + A) u | \tau_{-t} \varphi \rangle_{H_\varrho(\mathbb{R}; H^{-1}(|A|+1)) \times H_\varrho(\mathbb{R}; H^1(|A|+1))} \\
&= 0,
\end{aligned}$$

by Proposition 3.2.6, since $\tau_{-t} \varphi \in C_c^\infty(\mathbb{R}_{>0}; D(A^*))$. Hence, (3.10) holds and thus, employing the causality of $M(\partial_{0,\varrho})$ we derive

$$\begin{aligned}
& (\partial_{0,\varrho} M(\partial_{0,\varrho}) + A) \tau_t u \\
&= \chi_{\mathbb{R}_{\leq 0}}(\mathbf{m})(\partial_{0,\varrho} M(\partial_{0,\varrho}) + A) \tau_t u \\
&= \chi_{\mathbb{R}_{\leq 0}}(\mathbf{m}) \partial_{0,\varrho} M(\partial_{0,\varrho}) \tau_t u + A \tilde{g} \\
&= \partial_{0,\varrho} \chi_{\mathbb{R}_{\leq 0}}(\mathbf{m}) M(\partial_{0,\varrho}) \tau_t u + \delta_0 (M(\partial_{0,\varrho}) \tau_t u) (0-) + A \tilde{g} \\
&= \partial_{0,\varrho} \chi_{\mathbb{R}_{\leq 0}}(\mathbf{m}) M(\partial_{0,\varrho}) \tilde{g} + \delta_0 (M(\partial_{0,\varrho}) \tilde{g}) (0-) + A \tilde{g} \\
&= Q_0 \partial_{0,\varrho} M(\partial_{0,\varrho}) \tilde{g} + A \tilde{g},
\end{aligned}$$

where we implicitly have shown that $\partial_{0,\varrho} M(\partial_{0,\varrho}) \tilde{g} \in D(Q_0)$. Since $\tau_t u = \tilde{v} + \tilde{g}$ we infer

$$\partial_{0,\varrho} M(\partial_{0,\varrho}) \tau_t u + A \tilde{v} = Q_0 \partial_{0,\varrho} M(\partial_{0,\varrho}) \tilde{g}.$$

Now, we note that $\partial_{0,\varrho} M(\partial_{0,\varrho}) \tilde{g} \in D(P_0)$ by Lemma 3.2.4 and hence, Proposition 3.1.16 (d) together with the equality above yields

$$\begin{aligned}
\partial_{0,\varrho} M(\partial_{0,\varrho}) \tilde{v} &= \partial_{0,\varrho} M(\partial_{0,\varrho}) (\tau_t u - \tilde{g}) \\
&= \partial_{0,\varrho} M(\partial_{0,\varrho}) \tau_t u - (K_\varrho \tilde{g} + Q_0 \partial_{0,\varrho} M(\partial_{0,\varrho}) \tilde{g}) - \\
&\quad - \delta_0 ((M(\partial_{0,\varrho}) \tilde{g}) (0+) - (M(\partial_{0,\varrho}) \tilde{g}) (0-)) \\
&= -A \tilde{v} - K_\varrho \tilde{g} + \delta_0 ((M(\partial_{0,\varrho}) \tilde{g}) (0-) - (M(\partial_{0,\varrho}) \tilde{g}) (0+)).
\end{aligned}$$

Hence,

$$(\partial_{0,\varrho} M(\partial_{0,\varrho}) + A) \tilde{v} = -K_\varrho \tilde{g} + \delta_0 ((M(\partial_{0,\varrho}) \tilde{g}) (0-) - (M(\partial_{0,\varrho}) \tilde{g}) (0+)).$$

The latter gives $\tilde{g} \in \text{His}_\varrho(M, A)$. Indeed, we have that

$$\begin{aligned}
& S_\varrho (-K_\varrho \tilde{g} + \delta_0 ((M(\partial_{0,\varrho}) \tilde{g}) (0-) - (M(\partial_{0,\varrho}) \tilde{g}) (0+))) - \chi_{\mathbb{R}_{\geq 0}} \tilde{g} (0-) \\
&= \tilde{v} - \chi_{\mathbb{R}_{\geq 0}} \tilde{g} (0-) \in H_\varrho^1(\mathbb{R}; H),
\end{aligned}$$

since $\tilde{g}(0-) = \tilde{v}(0+)$ and $\tilde{v} \in \chi_{\mathbb{R}_{\geq 0}}(\mathbf{m})[H_\varrho^1(\mathbb{R}; H)]$. Hence, we also have that

$$\tilde{v} = S_\varrho \left(\delta_0 \Gamma_{(M,A)}^\varrho \tilde{g} - K_\varrho \tilde{g} \right).$$

□

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We are now able to define a semigroup associated with M and A .

Definition. Let $\varrho > \max\{s_0(M, A), b(M)\}$ and $g \in \text{His}_\varrho(M, A)$. Set

$$v := S_\varrho \left(\delta_0 \Gamma_{(M,A)}^\varrho g - K_\varrho g \right)$$

and $u := v + g$. Then, for $t \geq 0$ we define the operator

$$T_{(M,A)}(t) : D_\varrho \subseteq \text{IV}_\varrho(M, A) \times \text{His}_\varrho(M, A) \rightarrow \text{IV}_\varrho(M, A) \times \text{His}_\varrho(M, A)$$

with

$$D_\varrho := \{(g(0-), g) ; g \in \text{His}_\varrho(M, A)\}$$

by

$$T_{(M,A)}(t)(g(0-), g) := (v(t+), \chi_{\mathbb{R}_{\leq 0}}(\mathbf{m})\tau_t u) = (\tau_t u(0), \chi_{\mathbb{R}_{\leq 0}}(\mathbf{m})\tau_t u).$$

Moreover we set

$$\begin{aligned} T_{(M,A)}^{(1)}(t)(g(0-), g) &:= v(t+) \\ T_{(M,A)}^{(2)}(t)(g(0-), g) &:= \chi_{\mathbb{R}_{\leq 0}}(\mathbf{m})\tau_t u. \end{aligned}$$

Remark 3.3.2. Of course, both operator families $T_{(M,A)}^{(1)}$ and $T_{(M,A)}^{(2)}$ can be defined in terms of the other one. In fact, we have that

$$T_{(M,A)}^{(1)}(t)(g(0-), g) = \left(T_{(M,A)}^{(2)}(t)(g(0-), g) \right) (0-)$$

and

$$\left(T_{(M,A)}^{(2)}(t)(g(0-), g) \right) (s) = \begin{cases} g(t+s) & \text{if } s \leq -t, \\ T_{(M,A)}^{(1)}(t+s)(g(0-), g) & \text{if } -t < s \leq 0, \end{cases} \quad (s \leq 0).$$

We first show that $T_{(M,A)}$ satisfies the properties of a C_0 -semigroup with respect to the topology on $H \times H_\mu(\mathbb{R}_{\leq 0}; H)$ for each $\mu \leq \varrho$.

Proposition 3.3.3. *Let $\varrho > \max\{s_0(M, A), b(M)\}$, $t \geq 0$ and $T_{(M,A)}(t)$ like above. Then, $T_{(M,A)}(t)$ is linear and we have that*

$$T_{(M,A)}(t)(g(0-), g) \rightarrow T_{(M,A)}(0)(g(0-), g) = (g(0-), g)$$

in $H \times H_\mu(\mathbb{R}_{\leq 0}; H)$ for all $g \in \text{His}_\varrho(M, A)$ and $\mu \leq \varrho$ as $t \rightarrow 0$ and

$$T_{(M,A)}(t+s) = T_{(M,A)}(t)T_{(M,A)}(s)$$

for each $t, s \geq 0$. In particular, $T_{(M,A)}(\cdot)(g(0-), g) : \mathbb{R}_{\geq 0} \rightarrow H \times H_\mu(\mathbb{R}_{\leq 0}; H)$ is continuous for each $g \in \text{His}_\varrho(M, A)$ and $\mu \leq \varrho$.

Proof. The linearity of $T_{(M,A)}(t)$ is obvious since all operators involved are linear. Moreover, we have that

$$T_{(M,A)}(0)(g(0-), g) = (v(0+), \chi_{\mathbb{R}_{\leq 0}}(\mathbf{m})u) = (g(0-), g)$$

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for $g \in \text{His}_\varrho(M, A)$ by Proposition 3.2.6 and

$$\begin{aligned} & |T_{(M,A)}(t)(g(0-), g) - (g(0-), g)|_{H \times H_\varrho(\mathbb{R}_{\leq 0}; H)}^2 \\ &= |v(t+) - g(0-)|_H^2 + |\chi_{\mathbb{R}_{\leq 0}}(\mathfrak{m})\tau_t u - \chi_{\mathbb{R}_{\leq 0}}(\mathfrak{m})u|_{H_\varrho(\mathbb{R}; H)}, \\ &\leq |v(t+) - g(0-)|_H^2 + |\tau_t u - u|_{H_\varrho(\mathbb{R}; H)} \rightarrow 0 \quad (t \rightarrow 0) \end{aligned}$$

by right continuity of v , $v(0+) = g(0-)$ and the strong continuity of $t \mapsto \tau_t \in L(H_\varrho(\mathbb{R}; H))$. Moreover, since $H_\varrho(\mathbb{R}_{\leq 0}; H) \hookrightarrow H_\mu(\mathbb{R}_{\leq 0}; H)$ for $\mu \leq \varrho$, the convergence also holds in $H \times H_\mu(\mathbb{R}_{\leq 0}; H)$. Let now $t, s \geq 0$ and $g \in \text{His}_\varrho(M, A)$ and u like above. Then we have by Proposition 3.3.1

$$\chi_{\mathbb{R}_{\geq 0}}(\mathfrak{m})\tau_s u = S_\varrho \left(\delta_0 \Gamma_{(M,A)}^\varrho \chi_{\mathbb{R}_{\leq 0}}(\mathfrak{m})\tau_s u - K_\varrho \chi_{\mathbb{R}_{\leq 0}}(\mathfrak{m})\tau_s u \right)$$

and thus,

$$T_{(M,A)}(t)T_{(M,A)}(s)(g(0-), g) = (\tau_{t+s}u(0), \chi_{\mathbb{R}_{\leq 0}}(\mathfrak{m})\tau_{t+s}u) = T_{(M,A)}(t+s)g. \quad \square$$

The question which arises now is: can we extend the so-defined $T_{(M,A)}$ to a C_0 -semigroup on

$$X_\varrho^\mu := \overline{D}^{H \times H_\mu(\mathbb{R}_{\leq 0}; H)}$$

for some $\mu \leq \varrho$? Before we can answer this question, we need the following pre-requisites.

Lemma 3.3.4. *Let $\varrho > \max\{s_0(M, A), b(M)\}$ and $g \in \text{His}_\varrho(M, A)$. We define*

$$r_g : \mathbb{R}_{>\varrho} \rightarrow H$$

by

$$r_g(\lambda) := (\lambda M(\lambda) + A)^{-1} \left(\frac{1}{\sqrt{2\pi}} \Gamma_{(M,A)}^\varrho g - (\mathcal{L}_\lambda K_\varrho g)(0) \right) \quad (\lambda > \varrho).$$

Then $r_g \in C^\infty(\mathbb{R}_{>\varrho}; H)$. More precisely

$$r_g(\lambda) = \mathcal{L}_\lambda \left(T_{(M,A)}^{(1)}(\cdot)(g(0-), g) \right)(0) \quad (\lambda > \varrho).$$

Proof. We set

$$v := S_\varrho \left(\delta_0 \Gamma_{(M,A)}^\varrho g - K_\varrho g \right) = T_{(M,A)}^{(1)}(\cdot)(g(0-), g).$$

By definition of $\text{His}_\varrho(M, A)$ we know that $v \in H_\varrho(\mathbb{R}_{\geq 0}; H)$. Thus, by Corollary A.8 we have that $\widehat{v} \in \mathcal{H}^2(\mathbb{C}_{\text{Re}>\varrho}; H)$ and so, in particular, $\widehat{v}|_{\mathbb{R}_{>\varrho}} \in C^\infty(\mathbb{R}_{>\varrho}; H)$. For $\lambda > \varrho$ we compute

$$\begin{aligned} \widehat{v}(\lambda) &= (\mathcal{L}_\lambda v)(0) \\ &= \left(\mathcal{L}_\lambda S_\varrho \left(\delta_0 \Gamma_{(M,A)}^\varrho g - K_\varrho g \right) \right)(0). \end{aligned}$$

Using Proposition 3.1.13 we get that

$$S_\varrho \left(\delta_0 \Gamma_{(M,A)}^\varrho g - K_\varrho g \right) \sim S_\lambda \left(\delta_0 \Gamma_{(M,A)}^\varrho g - K_\varrho g \right)$$

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and thus,

$$\begin{aligned}\widehat{v}(\lambda) &= (\lambda M(\lambda) + A)^{-1} \left(\mathcal{L}_\lambda \left(\delta_0 \Gamma_{(M,A)}^\varrho g - K_\varrho g \right) \right) (0) \\ &= (\lambda M(\lambda) + A)^{-1} \left(\frac{1}{\sqrt{2\pi}} \Gamma_{(M,A)}^\varrho g - (\mathcal{L}_\lambda K_\varrho g) (0) \right),\end{aligned}$$

which yields the assertion. \square

Lemma 3.3.5. *Let $\varrho > \max\{s_0(M, A), b(M)\}$. Assume that for some $\mu \leq \varrho$ and $\omega \geq \varrho$ we have that*

$$T_{(M,A)}^{(1)} : D_\varrho \subseteq X_\varrho^\mu \rightarrow C_\omega(\mathbb{R}_{\geq 0}; H)$$

is bounded. Then for each $\varepsilon > 0$ the operator

$$T_{(M,A)}^{(2)} : D_\varrho \subseteq X_\varrho^\mu \rightarrow C_{\omega+\varepsilon}(\mathbb{R}_{\geq 0}; H_\mu(\mathbb{R}_{\leq 0}; H))$$

is bounded.

Proof. Let $\varepsilon > 0$ and $g \in \text{His}_\varrho(M, A)$. Then by Remark 3.3.2 we have that

$$\begin{aligned}& \left| T_{(M,A)}^{(2)}(t)(g(0-), g) \right|_{H_\mu(\mathbb{R}_{\leq 0}; H)}^2 \\ &= \int_{-\infty}^{-t} |g(t+s)|_H^2 e^{-2\mu s} \, ds + \int_{-t}^0 |T_{(M,A)}^{(1)}(t+s)(g(0-), g)|_H^2 e^{-2\mu s} \, ds \\ &= \left(\int_{-\infty}^0 |g(s)|_H^2 e^{-2\mu s} \, ds + \int_0^t |T_{(M,A)}^{(1)}(s)(g(0-), g)|_H^2 e^{-2\mu s} \, ds \right) e^{2\mu t} \\ &\leq \left(|g|_{H_\mu(\mathbb{R}_{\leq 0}; H)}^2 + C |(g(0-), g)|_{X_\varrho^\mu}^2 \int_0^t e^{2(\omega-\mu)s} \, ds \right) e^{2\mu t} \\ &\leq e^{2\mu t} |g|_{H_\mu(\mathbb{R}_{\leq 0}; H)}^2 + C |(g(0-), g)|_{X_\varrho^\mu}^2 t e^{2\omega t} \\ &\leq \tilde{C} e^{2(\omega+\varepsilon)t} |(g(0-), g)|_{X_\varrho^\mu}^2\end{aligned}$$

for all $t \geq 0$, where $\tilde{C} := 1 + C \frac{1}{2\varepsilon e}$. Using that $T_{(M,A)}^{(2)}(\cdot)(g(0-), g) : \mathbb{R}_{\geq 0} \rightarrow H_\mu(\mathbb{R}_{\leq 0}; H)$ is continuous by Proposition 3.3.3, we have shown that

$$T_{(M,A)}^{(2)} : D_\varrho \subseteq X_\varrho^\mu \rightarrow C_{\omega+\varepsilon}(\mathbb{R}_{\geq 0}; H_\mu(\mathbb{R}_{\leq 0}; H))$$

is bounded. \square

We can now prove a Hille-Yosida type result for the semigroup $T_{(M,A)}$.

Theorem 3.3.6. *Let $\varrho > \max\{s_0(M, A), b(M)\}$ and consider for $t \geq 0$ the mapping $T_{(M,A)}(t) : D_\varrho \subseteq H \times H_\varrho(\mathbb{R}_{\leq 0}; H) \rightarrow H \times H_\varrho(\mathbb{R}_{\leq 0}; H)$. For $g \in \text{His}_\varrho(M, A)$ we define*

$$r_g : \mathbb{R}_{>\varrho} \rightarrow H$$

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by

$$r_g(\lambda) := (\lambda M(\lambda) + A)^{-1} \left(\frac{1}{\sqrt{2\pi}} \Gamma_{(M,A)}^\varrho g - (\mathcal{L}_\lambda K_\varrho g)(0) \right) \quad (\lambda > \varrho).$$

Then $T_{(M,A)}$ can be extended to a C_0 -semigroup on $X_\varrho^\mu := \overline{D_\varrho}^{H \times H_\mu(\mathbb{R}_{\leq 0}; H)}$ for some $\mu \leq \varrho$ if and only if there is $M \geq 1, \omega \geq \varrho$ such that

$$\frac{1}{n!} |r_g^{(n)}(\lambda)| \leq \frac{M}{(\lambda - \omega)^{n+1}} \left(|g(0-)| + |g|_{H_\mu(\mathbb{R}_{\leq 0}; H)} \right) \quad (n \in \mathbb{N}, \lambda > \omega).$$

In this case we have that

$$\begin{aligned} T_{(M,A)}^{(1)} : X_\varrho^\mu &\rightarrow C_\omega(\mathbb{R}_{\geq 0}; H) \\ T_{(M,A)}^{(2)} : X_\varrho^\mu &\rightarrow C_{\omega+\varepsilon}(\mathbb{R}_{\geq 0}; H_\mu(\mathbb{R}_{\leq 0}; H)) \end{aligned}$$

are bounded for each $\varepsilon > 0$.

Proof. First assume that we can extend $T_{(M,A)}$ to a C_0 -semigroup on X_ϱ^μ for some $\mu \leq \varrho$. Then there exists $M \geq 1$ and $\omega \in \mathbb{R}$ such that

$$|T_{(M,A)}(t)(g(0-), g)|_{H \times H_\mu(\mathbb{R}_{\leq 0}; H)} \leq M e^{\omega t} |(g(0-), g)|_{H \times H_\mu(\mathbb{R}_{\leq 0}; H)} \quad (t \geq 0)$$

for all $g \in \text{His}_\varrho(M, A)$. In particular, we have that

$$v := T_{(M,A)}^{(1)}(\cdot)(g(0-), g)$$

satisfies

$$|v(t)| \leq M e^{\omega t} |(g(0-), g)|_{H \times H_\mu(\mathbb{R}_{\leq 0}; H)}.$$

Using Lemma 3.3.4 we derive

$$\begin{aligned} r_g(\lambda) &= (\mathcal{L}_\lambda v)(0) \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\lambda s} v(s) \, ds \\ &= \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-\lambda s} v(s) \, ds \end{aligned}$$

for each $\lambda > \max\{\omega, \varrho\}$ and consequently

$$r_g^{(n)}(\lambda) = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-\lambda s} (-s)^n v(s) \, ds.$$

Hence, we have that

$$|r_g^{(n)}(\lambda)| \leq \frac{1}{\sqrt{2\pi}} M |(g(0-), g)|_{H \times H_\mu(\mathbb{R}_{\leq 0}; H)} \int_0^\infty e^{(\omega-\lambda)s} s^n \, ds$$

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$$\begin{aligned}
&= \frac{1}{\sqrt{2\pi}} M |(g(0-), g)|_{H \times H_\mu(\mathbb{R}_{\leq 0}; H)} n! \frac{1}{(\lambda - \omega)^n} \int_0^\infty e^{(\omega - \lambda)s} \, ds \\
&= \frac{1}{\sqrt{2\pi}} M |(g(0-), g)|_{H \times H_\mu(\mathbb{R}_{\leq 0}; H)} \frac{n!}{(\lambda - \omega)^{n+1}},
\end{aligned}$$

which gives the desired estimate. Assume now that there is $M \geq 1, \omega \geq \varrho$ such that

$$\frac{1}{n!} |r_g^{(n)}(\lambda)| \leq \frac{M}{(\lambda - \omega)^{n+1}} |(g(0-), g)|_{H \times H_\mu(\mathbb{R}_{\leq 0}; H)} \quad (\lambda > \omega).$$

Then, the function $\tilde{r} : \mathbb{R}_{>0} \rightarrow H$ with $\tilde{r}(\lambda) = r(\lambda + \omega)$ satisfies the conditions of the Widder-Arendt Theorem C.6 and thus, there is $f \in L_\infty(\mathbb{R}_{\geq 0}; H)$ with $|f|_{L_\infty} \leq M |(g(0-), g)|_{H \times H_\mu(\mathbb{R}_{\leq 0}; H)}$ such that

$$\tilde{r}_g(\lambda) = \int_0^\infty e^{-\lambda s} f(s) \, ds \quad (\lambda > 0).$$

Hence, we have

$$\hat{v}(\lambda) = r_g(\lambda) = \tilde{r}_g(\lambda - \omega) = \int_0^\infty e^{-(\lambda - \omega)s} f(s) \, ds \quad (\lambda > \omega)$$

and by analytic continuation, we infer that

$$\hat{v}(z) = \int_0^\infty e^{-zs} e^{\omega s} f(s) \, ds \quad (z \in \mathbb{C}_{\operatorname{Re} > \omega}).$$

By the injectivity of the Fourier-Laplace transform, we derive

$$v(t) = \sqrt{2\pi} e^{\omega t} f(t) \quad (t \in \mathbb{R})$$

and so,

$$|v(t)|_H = \sqrt{2\pi} e^{\omega t} |f(t)|_H \leq \sqrt{2\pi} M e^{\omega t} |(g(0-), g)|_{H \times H_\mu(\mathbb{R}_{\leq 0}; H)}.$$

In other words, we have that

$$T_{(M,A)}^{(1)} : D_\varrho \subseteq X_\varrho^\mu \rightarrow C_\omega(\mathbb{R}; H)$$

is bounded and thus, can be extended to a bounded operator defined on X_ϱ^μ . Moreover, for each $\varepsilon > 0$ we have by Lemma 3.3.5

$$T_{(M,A)}^{(2)} : D_\varrho \subseteq X_\varrho^\mu \rightarrow C_{\omega+\varepsilon}(\mathbb{R}_{\geq 0}; H_\mu(\mathbb{R}_{\leq 0}; H))$$

is bounded and hence, it can be extended to X_ϱ^μ . Summarizing, we have shown that $T_{(M,A)}$ can be extended to a C_0 -semigroup on X_ϱ^μ . \square

We conclude this section by a result which relates the classical exponential stability of $T_{(M,A)}$ with the exponential stability of the corresponding evolutionary problem in the sense of Chap-

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ter 2. More precisely, we show that for a certain class of material laws, the growth bound of the semigroup $T_{(M,A)}$ can be estimated by $s_0(M, A)$, which would yield the exponential stability of the semigroup if the evolutionary problem is exponentially stable by Theorem 2.1.3. This theorem can be seen as a generalization of the well-known Gearhart-Prüß Theorem and its proof indeed follows the lines of the proof of the classical version presented in [ABHN11, Theorem 5.2.1].

Theorem 3.3.7. *Let $\varrho > \max\{s_0(M, A), b(M)\}$ and $s_0(M, A) < \mu \leq \varrho$. We assume that $T_{(M,A)}$ defines a C_0 -semigroup on X_ϱ^μ , that $\mathbb{C}_{\operatorname{Re}>\mu} \setminus D(M)$ is discrete and for each $\lambda > 0$ the mapping*

$$\Phi_\lambda : \mathbb{C}_{\operatorname{Re}>\mu} \cap D(M) \ni z \mapsto (zM(z) - (z + \lambda)M(z + \lambda)) \in L(H)$$

is bounded. Moreover, assume that

$$K_\varrho : \operatorname{His}_\varrho(M, A) \subseteq H_\varrho(\mathbb{R}_{\leq 0}; H) \rightarrow H_\mu(\mathbb{R}_{\geq 0}; H)$$

is well-defined and bounded. Then $\omega(T_{(M,A)}) \leq \mu$, where $\omega(T_{(M,A)})$ denotes the growth bound of $T_{(M,A)}$.

Proof. By Datko's Lemma (see [Dat70] or [EN00, Chapter V, Theorem 1.8]) it suffices to prove

$$\int_0^\infty |T_{(M,A)}(t)(x, f)|_{X_\varrho^\mu}^2 e^{-2(\mu+\varepsilon)t} dt < \infty \quad (3.11)$$

for each $(x, f) \in X_\varrho^\mu$ and each $\varepsilon > 0$. For doing so, let $\varepsilon > 0$. We define

$$\begin{aligned} S : \mathbb{C}_{\operatorname{Re}>\mu} &\rightarrow L(H) \\ z &\mapsto (zM(z) + A)^{-1} \end{aligned}$$

which is analytic and bounded since $\mu > s_0(M, A)$. Let $g \in \operatorname{His}_\varrho(M, A)$. As $T_{(M,A)}$ is a C_0 -semigroup on X_ϱ^μ there exists $\omega \geq \varrho$ such that

$$T_{(M,A)}^{(1)} : X_\varrho^\mu \rightarrow C_\omega(\mathbb{R}_{\geq 0}; H)$$

is continuous. Hence, we have $T_{(M,A)}^{(1)}(\cdot)(g(0-), g) = S_\varrho \left(\delta_0 \Gamma_{(M,A)}^\varrho g - K_\varrho g \right) \in C_\omega(\mathbb{R}_{\geq 0}; H) \hookrightarrow H_{\omega+1}(\mathbb{R}_{\geq 0}; H)$ and thus,

$$\left(z \mapsto S(z) \left(\frac{1}{\sqrt{2\pi}} \Gamma_{(M,A)}^\varrho g - \widehat{K_\varrho g}(z) \right) \right) \in \mathcal{H}^2(\mathbb{C}_{\operatorname{Re}>\omega+1}; H)$$

with

$$\begin{aligned} \left\| S(\cdot) \left(\frac{1}{\sqrt{2\pi}} \Gamma_{(M,A)}^\varrho g - \widehat{K_\varrho g} \right) \right\|_{\mathcal{H}^2(\mathbb{C}_{\operatorname{Re}>\omega+1}; H)} &= \|T_{(M,A)}^{(1)}(\cdot)(g(0-), g)\|_{H_{\omega+1}(\mathbb{R}_{\geq 0}; H)} \\ &\leq C \| (g(0-), g) \|_{X_\varrho^\mu} \end{aligned}$$

for some $C > 0$ by the Theorem of Paley-Wiener (see Corollary A.8). Moreover, since $S \in \mathcal{H}^\infty(\mathbb{C}_{\operatorname{Re}>\mu}; L(H))$ and $\widehat{K_\varrho g} \in \mathcal{H}^2(\mathbb{C}_{\operatorname{Re}>\mu}; H)$ (since $K_\varrho g \in H_\mu(\mathbb{R}_{\geq 0}; H)$ by hypothesis), we

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infer that

$$\left(z \mapsto S(z) \widehat{K_\varrho g}(z) \right) \in \mathcal{H}^2(\mathbb{C}_{\text{Re} > \mu}; H) \subseteq \mathcal{H}^2(\mathbb{C}_{\text{Re} > \omega + 1}; H)$$

and thus,

$$\left(z \mapsto S(z) \Gamma_{(M,A)}^\varrho g \right) \in \mathcal{H}^2(\mathbb{C}_{\text{Re} > \omega + 1}; H).$$

We estimate for each $\mu < \kappa \leq \omega + 1$ and $\lambda > 0$

$$\begin{aligned} & \left(\int_{\mathbb{R}} \left| S(it + \kappa) \Gamma_{(M,A)}^\varrho g \right|^2 dt \right)^{\frac{1}{2}} \\ & \leq \left(\int_{\mathbb{R}} \left| S(it + \kappa + \lambda) \Gamma_{(M,A)}^\varrho g \right|^2 dt \right)^{\frac{1}{2}} + \\ & \quad + \left(\int_{\mathbb{R}} \left| (S(it + \kappa) - S(it + \kappa + \lambda)) \Gamma_{(M,A)}^\varrho g \right|^2 dt \right)^{\frac{1}{2}}. \end{aligned}$$

Using

$$\begin{aligned} & S(it + \kappa) - S(it + \kappa + \lambda) \\ & = ((it + \kappa) M(it + \kappa) + A)^{-1} - ((it + \kappa + \lambda) M(it + \kappa + \lambda) + A)^{-1} \\ & = -((it + \kappa) M(it + \kappa) + A)^{-1} \Phi_\lambda(it + \kappa) ((it + \kappa + \lambda) M(it + \kappa + \lambda) + A)^{-1} \\ & = -S(it + \kappa) \Phi_\lambda(it + \kappa) S(it + \kappa + \lambda) \end{aligned}$$

for each $t \in \mathbb{R}$, we can estimate the last integral by

$$\begin{aligned} & \left(\int_{\mathbb{R}} \left| (S(it + \kappa) - S(it + \kappa + \lambda)) \Gamma_{(M,A)}^\varrho g \right|^2 dt \right)^{\frac{1}{2}} \\ & \leq \|S\|_{\mathcal{H}^\infty(\mathbb{C}_{\text{Re} > \mu}; L(H))} \|\Phi_\lambda\|_\infty \left(\int_{\mathbb{R}} \left| S(it + \kappa + \lambda) \Gamma_{(M,A)}^\varrho g \right|^2 dt \right)^{\frac{1}{2}} \end{aligned}$$

and hence, for $\lambda > \omega + 1 - \mu$

$$\left(\int_{\mathbb{R}} \left| S(it + \kappa) \Gamma_{(M,A)}^\varrho g \right|^2 dt \right)^{\frac{1}{2}} \leq \tilde{C} |S(\cdot) \Gamma_{(M,A)}^\varrho g|_{\mathcal{H}^2(\mathbb{C}_{\text{Re} > \omega + 1}; H)},$$

with $\tilde{C} := 1 + \|S\|_{\mathcal{H}^\infty(\mathbb{C}_{\text{Re} > \mu}; L(H))} \|\Phi_\lambda\|_\infty$. Since κ was arbitrary, we infer $S(\cdot) \Gamma_{(M,A)}^\varrho g \in \mathcal{H}^2(\mathbb{C}_{\text{Re} > \mu}; H)$ and thus,

$$\left(z \mapsto S(z) \left(\frac{1}{\sqrt{2\pi}} \Gamma_{(M,A)}^\varrho g - \widehat{K_\varrho g}(z) \right) \right) \in \mathcal{H}^2(\mathbb{C}_{\text{Re} > \mu}; H).$$

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Again, by Paley-Wiener we have that $T_{(M,A)}^{(1)}(\cdot)(g(0-), g) \in H_\mu(\mathbb{R}_{\geq 0}; H)$ with

$$\begin{aligned}
& \left| T_{(M,A)}^{(1)}(\cdot)(g(0-), g) \right|_{H_\mu(\mathbb{R}_{\geq 0}; H)} \\
&= \left| S(\cdot) \left(\frac{1}{\sqrt{2\pi}} \Gamma_{(M,A)}^\varrho g - \widehat{K_\varrho g} \right) \right|_{\mathcal{H}^2(\mathbb{C}_{\text{Re} > \mu}; H)} \\
&\leq \frac{1}{\sqrt{2\pi}} \left| S(\cdot) \Gamma_{(M,A)}^\varrho g \right|_{\mathcal{H}^2(\mathbb{C}_{\text{Re} > \mu}; H)} + \left| S(\cdot) \widehat{K_\varrho g} \right|_{\mathcal{H}^2(\mathbb{C}_{\text{Re} > \mu}; H)} \\
&\leq \frac{1}{\sqrt{2\pi}} \tilde{C} \left| S(\cdot) \Gamma_{(M,A)}^\varrho g \right|_{\mathcal{H}^2(\mathbb{C}_{\text{Re} > \omega+1}; H)} + \|S\|_{\mathcal{H}^\infty(\mathbb{C}_{\text{Re} > \mu}; L(H))} \|K_\varrho g\|_{H_\mu(\mathbb{R}_{\geq 0}; H)} \\
&\leq \frac{1}{\sqrt{2\pi}} \tilde{C} \left| S(\cdot) \left(\Gamma_{(M,A)}^\varrho g - \widehat{K_\varrho g} \right) \right|_{\mathcal{H}^2(\mathbb{C}_{\text{Re} > \omega+1}; H)} + \\
&\quad + \left(\frac{\tilde{C}}{\sqrt{2\pi}} + 1 \right) \|S\|_{\mathcal{H}^\infty(\mathbb{C}_{\text{Re} > \mu}; L(H))} \|K_\varrho\| \|g\|_{H_\varrho(\mathbb{R}_{\leq 0}; H)} \\
&\leq \left(\frac{\tilde{C}}{\sqrt{2\pi}} C + \left(\frac{\tilde{C}}{\sqrt{2\pi}} + 1 \right) \|S\|_{\mathcal{H}^\infty(\mathbb{C}_{\text{Re} > \mu}; L(H))} \|K_\varrho\| \right) |(g(0-), g)|_{H \times H_\varrho(\mathbb{R}_{\leq 0}; H)}
\end{aligned}$$

or in other words

$$T_{(M,A)}^{(1)} : D_\varrho \subseteq X_\varrho^\mu \rightarrow H_\mu(\mathbb{R}_{\geq 0}; H)$$

is well-defined and bounded. Thus, using Remark 3.3.2, we obtain

$$\begin{aligned}
& \int_0^\infty |T_{(M,A)}^{(2)}(t)(g(0-), g)|_{H_\mu(\mathbb{R}_{\leq 0}; H)}^2 e^{-2(\mu+\varepsilon)t} dt \\
&= \int_0^\infty \left(\int_{-\infty}^{-t} |g(t+s)|^2 e^{-2\mu s} ds + \int_{-t}^0 |T_{(M,A)}^{(1)}(t+s)(g(0-), g)|^2 e^{-2\mu s} ds \right) e^{-2(\mu+\varepsilon)t} dt \\
&= \frac{1}{2\varepsilon} \left(|g|_{H_\mu(\mathbb{R}_{\leq 0}; H)}^2 + |T_{(M,A)}^{(1)}(\cdot)(g(0-), g)|_{H_\mu(\mathbb{R}_{\geq 0}; H)}^2 \right) \\
&\leq C |(g(0-), g)|_{X_\varrho^\mu}^2
\end{aligned}$$

which proves the well-definedness and boundedness of

$$T_{(M,A)}^{(2)} : D_\varrho \subseteq X_\varrho^\mu \rightarrow H_{\mu+\varepsilon}(\mathbb{R}_{\geq 0}; H_\mu(\mathbb{R}_{\leq 0}; H)).$$

Thus,

$$T_{(M,A)} : D_\varrho \subseteq X_\varrho^\mu \rightarrow H_{\mu+\varepsilon}(\mathbb{R}_{\geq 0}; X_\varrho^\mu)$$

is bounded and hence, (3.11) follows by continuous extension. \square

3.4. Applications

In this section, we apply the results of the three previous sections to several examples. More precisely, we discuss how the results can be applied to differential-algebraic equations, delay

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equations and integro-differential equations. Throughout, we assume that $A : D(A) \subseteq H \rightarrow H$ is a densely defined closed linear operator and that the evolutionary problem associated with M and A is well-posed, where the material law M will be specified in each case. For each of these examples we need to verify certain conditions for the material law in order to apply our results. For the ease of readability we recall these assumptions in the following. First of all we recall the definition

$$b(M) := \inf\{\varrho \geq 0; \mathbb{C}_{\operatorname{Re} \geq \varrho} \subseteq D(M), M|_{\mathbb{C}_{\operatorname{Re} \geq \varrho}} \text{ bounded}\}$$

In order to define the space of admissible histories, we assume that M is regularizing, i.e.

$$(b(M) < \infty) \wedge (\forall \varrho > b(M), x \in H : (M(\partial_{0,\varrho})\chi_{\mathbb{R}_{\geq 0}}x)(0+)) \text{ exists.} \quad (\text{REG})$$

We fix $\varrho > b(M)$ and recall the definitions

$$\begin{aligned} K_\varrho : \chi_{\mathbb{R}_{\leq 0}} [H_\varrho^1(\mathbb{R}; H)] &\rightarrow H_\varrho^{-1}(\mathbb{R}; H) \\ g &\mapsto P_0 \partial_{0,\varrho} M(\partial_{0,\varrho})g \end{aligned}$$

and

$$\begin{aligned} \Gamma_{(M,A)}^\varrho : \operatorname{His}_\varrho(M, A) &\rightarrow H \\ g &\mapsto (M(\partial_{0,\varrho})g)(0-) - (M(\partial_{0,\varrho})g)(0+), \end{aligned}$$

where we remark that

$$\Gamma_{(M,A)}^\varrho g = (M(\partial_{0,\varrho})\chi_{\mathbb{R}_{\geq 0}}g(0-))(0+) \quad (g \in \operatorname{His}_\varrho(M, A))$$

by Lemma 3.2.5. Furthermore, we recall

$$\begin{aligned} r_g : \mathbb{R}_{>\varrho} &\rightarrow H \\ \lambda &\mapsto (\lambda M(\lambda) + A)^{-1} \left(\Gamma_{(M,A)}^\varrho g - (\mathcal{L}_\lambda K_\varrho g)(0) \right), \end{aligned}$$

where M satisfies (REG), $\varrho > \max\{s_0(M, A), b(M)\}$ and $g \in \operatorname{His}_\varrho(M, A)$. By Theorem 3.3.6, this functions needs to verify the Hille-Yosida type condition, i.e. there exist $M \geq 1, \omega > \varrho$ such that

$$\forall g \in \operatorname{His}_\varrho(M, A), n \in \mathbb{N}, \lambda > \omega : \frac{1}{n!} |r_g^{(n)}(\lambda)|_H \leq \frac{M}{(\lambda - \omega)^{n+1}} |(g(0-), g)|_{H \times H_\mu(\mathbb{R}_{\leq 0}; H)} \quad (\text{HY})$$

for some $\mu \leq \varrho$ in order to obtain a strongly continuous semigroup $T_{(M,A)}$ on the Hilbert space $X_\varrho^\mu := \overline{D_\varrho}^{H \times H_\mu(\mathbb{R}_{\leq 0}; H)}$, where

$$D_\varrho := \{(g(0-), g); g \in \operatorname{His}_\varrho(M, A)\}.$$

Finally, we recall the conditions for the generalized Gearhart-Prüß Theorem Theorem 3.3.7, which provides an upper bound for the growth-type of the semigroup on X_ϱ^μ of the form

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$\omega(T_{(M,A)}) \leq \mu$ for some $s_0(M, A) < \mu \leq \varrho$:

$$\Phi_\lambda : \mathbb{C}_{\text{Re} > \mu} \cap D(M) \ni z \mapsto (zM(z) - (z + \lambda)M(z + \lambda)) \in L(H) \text{ is bounded} \quad (\text{GP-1})$$

for each $\lambda > 0$ and

$$K_\varrho : \text{His}_\varrho(M, A) \subseteq H_\varrho(\mathbb{R}_{\leq 0}; H) \rightarrow H_\mu(\mathbb{R}_{\geq 0}; H) \text{ is well-defined and bounded.} \quad (\text{GP-2})$$

3.4.1. Differential-algebraic equations

According to Proposition 3.2.10 evolutionary problems without memory are precisely those, where the material law M is given by

$$M(z) := M_0 + z^{-1}M_1 \quad (z \in \mathbb{C} \setminus \{0\})$$

for some bounded linear operators $M_0, M_1 \in L(H)$. Of course, those operators satisfy (REG). Indeed, we have that $b(M) = 0$ and for each $\varrho > 0$

$$(M(\partial_{0,\varrho})\chi_{\mathbb{R}_{\geq 0}}x)(0+) = M_0x$$

and consequently,

$$\Gamma_{(M,A)}^\varrho g = M_0g(0-)$$

for all $g \in \text{His}_\varrho(M, A)$ with $\varrho > \max\{s_0(M, A), 0\}$, which is given by

$$\text{His}_\varrho(M, A) = \{g \in \chi_{\mathbb{R}_{\leq 0}}(\text{m}) [H_\varrho^1(\mathbb{R}; H)] ; S_\varrho(\delta_0 M_0 g(0-)) - \chi_{\mathbb{R}_{\geq 0}} g(0-) \in H_\varrho^1(\mathbb{R}; H)\}$$

according to Proposition 3.2.10. The latter gives

$$\text{IV}_\varrho(M, A) = \{u_0 \in H ; S_\varrho(\delta_0 M_0 u_0) - \chi_{\mathbb{R}_{\geq 0}} u_0 \in H_\varrho^1(\mathbb{R}; H)\}.$$

Proposition 3.4.1. *Assume that M_0 has closed range. Denote by $\iota_0 : N(M_0)^\perp \rightarrow H$ and by $\iota_1 : R(M_0) \rightarrow H$ the canonical embeddings of the orthogonal complement of the null-space and the range of M_0 , respectively. Then, $\iota_1^* M_0 \iota_0 : N(M_0)^\perp \rightarrow R(M_0)$ is boundedly invertible and*

$$\left\{ u_0 \in D(A) ; (M_1 + A)u_0 \in R(M_0), \iota_0(\iota_1^* M_0 \iota_0)^{-1} \iota_1^*(M_1 + A)u_0 \in D(A) \right\} \subseteq \text{IV}_\varrho(M, A)$$

for each $\varrho > \max\{s_0(M, A), 0\}$.

Proof. Note that $\iota_1^* M_0 \iota_0$ is a bijective closed operator. Thus, the bounded invertibility follows by the closed graph theorem. Let now $u_0 \in D(A)$ such that $(M_1 + A)u_0 \in R(M_0)$ and $\iota_0(\iota_1^* M_0 \iota_0)^{-1} \iota_1^*(M_1 + A)u_0 \in D(A)$. Then we have that

$$\begin{aligned} S_\varrho(\delta_0 M_0 u_0) - \chi_{\mathbb{R}_{\geq 0}} u_0 &= S_\varrho(\delta_0 M_0 u_0 - (\partial_{0,\varrho} M_0 + M_1 + A)\chi_{\mathbb{R}_{\geq 0}} u_0) \\ &= -S_\varrho((M_1 + A)\chi_{\mathbb{R}_{\geq 0}} u_0). \end{aligned}$$

Since $(M_1 + A)u_0 \in R(M_0)$ we have that

$$(M_1 + A)u_0 = M_0 \iota_0(\iota_1^* M_0 \iota_0)^{-1} \iota_1^*(M_1 + A)u_0.$$

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We set $x := \iota_0 (\iota_1^* M_0 \iota_0)^{-1} \iota_1^* (M_1 + A) u_0$ and derive,

$$\begin{aligned} \partial_{0,\varrho} (S_\varrho (\delta_0 M_0 u_0) - \chi_{\mathbb{R}_{\geq 0}} u_0) &= -S_\varrho (\partial_{0,\varrho} (M_1 + A) \chi_{\mathbb{R}_{\geq 0}} u_0) \\ &= -S_\varrho (\partial_{0,\varrho} M_0 \chi_{\mathbb{R}_{\geq 0}} x). \end{aligned}$$

Using that $x \in D(A)$ by assumption, we conclude

$$\partial_{0,\varrho} (S_\varrho (\delta_0 M_0 u_0) - \chi_{\mathbb{R}_{\geq 0}} u_0) = -\chi_{\mathbb{R}_{\geq 0}} x + S_\varrho (\chi_{\mathbb{R}_{\geq 0}} (M_1 + A) x) \in H_\varrho(\mathbb{R}; H),$$

which shows $u_0 \in \text{IV}_\varrho(M, A)$. □

Remark 3.4.2.

- (a) In the case of a classical evolution equation, i.e. in the case $M_0 = 1$ and $M_1 = 0$, the latter proposition yields that

$$D(A^2) \subseteq \text{IV}_\varrho(M, A).$$

In case of an ordinary differential-algebraic equation, i.e. $A = 0$ in our setting, we can even show more.

- (b) Another approach to obtain solutions for a class of differential-algebraic equations is used in [Weh15], where even nonlinear problems are considered. There, one applies a backward difference scheme to obtain a discretised version of the equation, which turns out to be well-posed. Then one shows that these approximating solutions converge to a solution of the original problem in a suitable sense. Using this approach, a condition for admissible initial values is derived, even in the case, when $R(M_0)$ is not closed.

Proposition 3.4.3. *Assume that M_0 has closed range. Then*

$$\{u_0 \in H ; M_1 u_0 \in R(M_0)\} = \text{IV}_\varrho(M, 0)$$

for each $\varrho > \max\{s_0(M, 0), 0\}$. In particular, $\text{IV}_\varrho(M, 0)$ is closed and independent of ϱ .

Proof. According to Proposition 3.4.1 we have that

$$\{u_0 \in H ; M_1 u_0 \in R(M_0)\} \subseteq \text{IV}_\varrho(M, 0).$$

Assume now $u_0 \in \text{IV}_\varrho(M, 0)$. Then

$$u := (\partial_{0,\varrho} M_0 + M_1)^{-1} \delta_0 M_0 u_0$$

satisfies

$$u - \chi_{\mathbb{R}_{\geq 0}} u_0 \in H_\varrho^1(\mathbb{R}; H),$$

by definition of $\text{IV}_\varrho(M, 0)$. Thus,

$$M_1 u = \partial_{0,\varrho} M_0 (u - \chi_{\mathbb{R}_{\geq 0}} u_0) = M_0 \partial_{0,\varrho} (u - \chi_{\mathbb{R}_{\geq 0}} u_0),$$

which implies

$$M_1 u(t) \in R(M_0) \quad (t \in \mathbb{R} \text{ a.e.}).$$

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Hence, there is a sequence $(t_n)_{n \in \mathbb{N}}$ in $\mathbb{R}_{>0}$ such that $t_n \rightarrow 0$ and $M_1 u(t_n) \in R(M_0)$ for each $n \in \mathbb{N}$. Since $u(t_n) \rightarrow u_0$ due to continuity, we infer

$$M_1 u_0 \in R(M_0)$$

by the closedness of $R(M_0)$. □

Example 3.4.4 (DAEs in finite dimensions). Assume that $H = \mathbb{C}^n$ and $M_0, M_1 \in \mathbb{C}^{n \times n}$ and that $s_0(M, 0) < \infty$ with $M(z) = M_0 + z^{-1}M_1$. By the previous proposition we know that

$$\text{IV}_\varrho(M, 0) = \{u_0 \in \mathbb{C}^n; M_1 u_0 \in R(M_0)\}$$

for each $\varrho \geq \max\{s_0(M, 0), 0\}$. The class of ordinary differential-algebraic equations in finite dimensions have been widely studied and the set of so-called *consistent initial values* is well-known and given as follows: By the Weierstraß normal form for the matrix pair (M_0, M_1) , there exist regular matrices $P, Q \in \mathbb{C}^{n \times n}$ such that

$$PM_0Q = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad PM_1Q = \begin{pmatrix} J & 0 \\ 0 & 1 \end{pmatrix},$$

where $J \in \mathbb{C}^{d \times d}$ is a Jordan matrix (see e.g. [KM06, Theorem 2.7] and note that $s_0(M, 0) < \infty$ implies $N = 0$). The set of consistent initial values is given by (see [KM06, Theorem 2.12])

$$B := \left\{ u_0 \in \mathbb{C}^n; Q^{-1}u_0 \in \mathbb{C}^d \times \{0\} \right\}.$$

We show that $B = \text{IV}_\varrho(M, 0)$. If $u_0 \in B$ then

$$M_1 u_0 = P^{-1}PM_1QQ^{-1}u_0 = P^{-1} \begin{pmatrix} J & 0 \\ 0 & 1 \end{pmatrix} Q^{-1}u_0 = P^{-1} \begin{pmatrix} Jy \\ 0 \end{pmatrix},$$

where $(y, 0) := Q^{-1}u_0 \in \mathbb{C}^d \times \{0\}$. Setting $x := Q \begin{pmatrix} Jy \\ 0 \end{pmatrix}$ we compute

$$M_0 x = P^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} Q^{-1}x = P^{-1} \begin{pmatrix} Jy \\ 0 \end{pmatrix} = M_1 u_0$$

and thus, $u_0 \in \text{IV}_\varrho(M, 0)$. Let now $u_0 \in \text{IV}_\varrho(M, 0)$. Then there is $x \in H$ such that

$$M_1 u_0 = M_0 x$$

or equivalently,

$$\begin{pmatrix} J & 0 \\ 0 & 1 \end{pmatrix} Q^{-1}u_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} Q^{-1}x,$$

which yields $u_0 \in B$.

In the special setting of differential-algebraic equations we are able to strengthen the Hille-Yosida type result Theorem 3.3.6 in the following way.

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Proposition 3.4.5. *Let $\varrho > \max\{s_0(M, A), 0\}$. Then the following statements are equivalent:*

- (i) *There exists $\mu \leq \varrho$ such that $T_{(M,A)}$ extends to a C_0 -semigroup on X_ϱ^μ .*
- (ii) *There exists $\mu \leq \varrho$ such that (HY) holds.*
- (iii) *There exist $M \geq 1$ and $\omega \geq \varrho$, such that for each $x \in \text{IV}_\varrho(M, A)$, $n \in \mathbb{N}$ we have*

$$\frac{1}{n!} |s_x^{(n)}(\lambda)|_H \leq \frac{M}{(\lambda - \omega)^{n+1}} |x|_H \quad (\lambda > \omega), \quad (3.12)$$

where

$$\begin{aligned} s_x : \mathbb{R}_{>\omega} &\rightarrow H \\ \lambda &\rightarrow (\lambda M_0 + M_1 + A)^{-1}(M_0 x). \end{aligned}$$

- (iv) *$T_{(M,A)}$ extends to a C_0 -semigroup on X_ϱ^μ for each $\mu \leq \varrho$.*
- (v) *$T_{(M,A)}^{(1)}(\cdot, 0)$ extends to a C_0 -semigroup on $\overline{\text{IV}_\varrho(M, A)}^H$.*

Proof. (i) \Rightarrow (ii) holds by Theorem 3.3.6.

(ii) \Rightarrow (iii): There exist $\mu \leq \varrho$, $M \geq 1$, $\omega \geq \varrho$ such that

$$\frac{1}{n!} |r_g^{(n)}(\lambda)|_H \leq \frac{M}{(\lambda - \omega)^{n+1}} |(g(0-), g)|_{X_\varrho^\mu} \quad (\lambda > \omega, n \in \mathbb{N})$$

for each $g \in \text{His}_\varrho(M, A)$. Let $x \in \text{IV}_\varrho(M, A)$ and define $g_k(t) := \chi_{[-\frac{1}{k}, 0]}(t)(kt + 1)x$ for $k \in \mathbb{N}, t \leq 0$. Then $g_k \in \text{His}_\varrho(M, A)$ by Proposition 3.2.10 and $g_k(0-) = x$. Recall that

$$r_{g_k}(\lambda) = (\lambda M_0 + M_1 + A)^{-1}(M_0 x) = s_x(\lambda) \quad (\lambda > \omega)$$

by Proposition 3.2.10 for each $k \in \mathbb{N}$. Thus, we have that

$$\frac{1}{n!} |s_x^{(n)}(\lambda)|_H = \lim_{k \rightarrow \infty} \frac{1}{n!} |r_{g_k}^{(n)}(\lambda)|_H \leq \lim_{k \rightarrow \infty} \frac{M}{(\lambda - \omega)^{n+1}} |(x, g_k)|_{X_\varrho^\mu} = \frac{M}{(\lambda - \omega)^{n+1}} |x|_H$$

for every $n \in \mathbb{N}$ and $\lambda > \omega$.

(iii) \Rightarrow (iv): Let $\mu \leq \varrho$. We recall that for each $g \in \text{His}_\varrho(M, A)$ we have that

$$r_g = s_{g(0-)}.$$

Thus,

$$\frac{1}{n!} |r_g^{(n)}(\lambda)|_H = \frac{1}{n!} |s_{g(0-)}^{(n)}(\lambda)|_H \leq \frac{M}{(\lambda - \omega)^{n+1}} |g(0-)|_H \leq \frac{M}{(\lambda - \omega)^{n+1}} |(g(0-), g)|_{H \times H_\mu(\mathbb{R}_{\leq 0}; H)}$$

and hence, the assertion follows from Theorem 3.3.6.

(iv) \Rightarrow (v): Let $\mu \leq \varrho$. We first prove that $(x, 0) \in X_\varrho^\mu$ for each $x \in \overline{\text{IV}_\varrho(M, A)}$. For doing so, let $x \in \overline{\text{IV}_\varrho(M, A)}$ and choose a sequence $(x_n)_{n \in \mathbb{N}}$ in $\text{IV}_\varrho(M, A)$ such that $x_n \rightarrow x$ as $n \rightarrow \infty$. Define $g_n(t) := \chi_{[-\frac{1}{k}, 0]}(t)(kt + 1)x_n$ for $n \in \mathbb{N}, t \leq 0$. Then $g_n \in \text{His}_\varrho(M, A)$ and thus, $(x_n, g_n) \in D_\varrho$. Since $g_n \rightarrow 0$ in $H_\mu(\mathbb{R}_{\leq 0}; H)$ we infer that $(x, 0) \in X_\varrho^\mu$. Hence, $T_{(M,A)}^{(1)}(\cdot, 0)$ is

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a well-defined semigroup on $\overline{\text{IV}_\varrho(M, A)}$, which is strongly continuous, since $T_{(M, A)}$ is strongly continuous.

(v) \Rightarrow (i): It suffices to prove that the operator $T_{(M, A)} : D_\varrho \subseteq X_\varrho^\mu \rightarrow C_\omega(\mathbb{R}_{\geq 0}; H \times H_\mu(\mathbb{R}_{\leq 0}; H))$ is bounded for some $\omega \in \mathbb{R}$. By assumption, we have that

$$T_{(M, A)}^{(1)}(\cdot, 0) : \overline{\text{IV}_\varrho(M, A)} \rightarrow C_{\omega'}(\mathbb{R}_{\geq 0}; H)$$

is bounded for some $\omega' \in \mathbb{R}$. We assume without loss of generality that $\omega' \geq \varrho$ and set $\omega := \omega' + 1$. Let $g \in \text{His}_\varrho(M, A)$ and recall that $K_\varrho g = 0$ by Proposition 3.2.10. Then we have that

$$T_{(M, A)}^{(1)}(\cdot)(g(0-), g) = S_\varrho \left(\delta_0 \Gamma_{(M, A)}^\varrho g - K_\varrho g \right) = S_\varrho(\delta_0 M_0 g(0-)) = T_{(M, A)}^{(1)}(\cdot)(g(0-), 0)$$

and thus,

$$T_{(M, A)}^{(1)} : D_\varrho \subseteq X_\varrho^\mu \rightarrow C_{\omega'}(\mathbb{R}_{\geq 0}; H)$$

is bounded. By Lemma 3.3.5 we infer that

$$T_{(M, A)}^{(2)} : D_\varrho \subseteq X_\varrho^\mu \rightarrow C_\omega(\mathbb{R}_{\geq 0}; H)$$

is bounded, which gives the assertion. \square

Remark 3.4.6. (a) An easy computation gives that

$$s_x^{(n)}(\lambda) = (-1)^n n! \left((\lambda M_0 + M_1 + A)^{-1} M_0 \right)^{n+1} x$$

for each $n \in \mathbb{N}, x \in H, \lambda \geq \varrho$, where s_x is given as in Proposition 3.4.5 (iii). Hence, (3.12) holds if and only if

$$\left| \left((\lambda M_0 + M_1 + A)^{-1} M_0 \right)^{n+1} x \right|_H \leq \frac{M}{(\lambda - \omega)^{n+1}} |x|_H \quad (\lambda > \omega, n \in \mathbb{N}, x \in \text{IV}_\varrho(M, A))$$

for some $M \geq 1, \omega \geq \varrho$.

(b) If $M_0 = 1, M_1 = 0$ the equivalence (iii) \Leftrightarrow (v) is the well-known Hille-Yosida Theorem (see e.g. [EN00, Ch. II, Th. 3.8]). Indeed, we have that

$$s_x(\lambda) = (\lambda + A)^{-1} x$$

for $x \in \text{IV}_\varrho(M, A)$ and hence, (iii) in Proposition 3.4.5 gives that

$$|(\lambda + A)^{-n} x|_H \leq \frac{M}{(\lambda - \omega)^n} |x|_H$$

for some $M \geq 1$ and $\omega \geq \varrho$. Using $D(A^2) \subseteq \text{IV}_\varrho(M, A)$ by Remark 3.4.2 we derive

$$\|(\lambda + A)^{-n}\| \leq \frac{M}{(\lambda - \omega)^n},$$

which is the classical Hille-Yosida condition for $-A$ to be a generator of a C_0 -semigroup

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which is nothing but $T_{(1,A)}^{(1)}(\cdot, 0)$ on $\overline{\text{IV}_\varrho(M, A)} = H$.

Proposition 3.4.7. *Let $\varrho > \max\{s_0(M, A), 0\}$ and assume that (HY) holds. Then $\omega(T_{(M,A)}) \leq s_0(M, A)$.*

Proof. Let $s_0(M, A) < \mu \leq \varrho$. We apply Theorem 3.3.7. First we note that $\mathbb{C}_{\text{Re} > \mu} \setminus D(M) \subseteq \{0\}$ and hence, it is discrete. Moreover, by Proposition 3.4.5 $T_{(M,A)}$ extends to a C_0 -semigroup on X_ϱ^μ . We have to show that (GP-1) and (GP-2) hold. This, however is clear, since

$$\Phi_\lambda(z) = zM(z) - (z + \lambda)M(z + \lambda) = -\lambda M_0$$

for each $z \in \mathbb{C}_{\text{Re} > \mu} \setminus \{0\}$, $\lambda > 0$ and hence, (GP-1) is satisfied. Furthermore, $K_\varrho = 0$ by Proposition 3.2.10 and hence, (GP-2) holds trivially. Thus, Theorem 3.3.7 is applicable and hence, $\omega(T_{(M,A)}) \leq \mu$. Since $\mu > s_0(M, A)$ was arbitrary, we infer $\omega(T_{(M,A)}) \leq s_0(M, A)$. \square

Remark 3.4.8. In the case $M_0 = 1, M_1 = 0$ we get $\omega(T) \leq s_0(1, A)$, which is the well-known Gearhart-Prüß Theorem for C_0 -semigroups on Hilbert spaces (see e.g. [Prü84] or [EN00, Ch. V, Th. 1.11]).

We conclude this subsection, by discussing an abstract version of the heat equation in the presented framework.

Example 3.4.9. Let H_0, H_1 be Hilbert spaces and $C : D(C) \subseteq H_0 \rightarrow H_1$ densely defined closed and linear. We consider the evolutionary equation

$$\left(\partial_{0,\varrho} \begin{pmatrix} \eta & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & k^{-1} \end{pmatrix} + \begin{pmatrix} 0 & -C^* \\ C & 0 \end{pmatrix} \right) \begin{pmatrix} \vartheta \\ q \end{pmatrix} = F,$$

where $\eta \in L(H_0), k \in L(H_1)$ are selfadjoint, strictly m -accretive operators with

$$\begin{aligned} \langle \eta x_0 | x_0 \rangle_{H_0} &\geq c_0 |x_0|_{H_0}^2, \\ \langle k x_1 | x_1 \rangle_{H_1} &\geq c_1 |x_1|_{H_1}^2 \end{aligned}$$

for some $c_0, c_1 > 0$ and each $(x_0, x_1) \in H_0 \oplus H_1 =: H$. Thus, in the setting of differential-algebraic equations we have

$$M_0 = \begin{pmatrix} \eta & 0 \\ 0 & 0 \end{pmatrix}, \quad M_1 = \begin{pmatrix} 0 & 0 \\ 0 & k^{-1} \end{pmatrix}, \quad A = \begin{pmatrix} 0 & -C^* \\ C & 0 \end{pmatrix}.$$

We note that the evolutionary problem is well-posed by Proposition 1.2.20. We have that $s_0(M, A) \leq 0$. Indeed, for $\varrho > 0$ and $z \in \mathbb{C}_{\text{Re} > \varrho}$ we get

$$\text{Re} \langle (zM_0 + M_1)x | x \rangle_H \geq \min \{ \varrho c_0, c_1^{-1} \} |x|_H^2$$

for each $x \in H$. Hence $(zM_0 + M_1)$ is strictly accretive and by Proposition 1.2.17, so is $zM_0 + M_1 + A$ and we get

$$\|(zM_0 + M_1 + A)^{-1}\| \leq \frac{1}{\min \{ \varrho c_0, c_1^{-1} \}}.$$

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This proves $s_0(M, A) \leq 0$. If C is boundedly invertible, we even obtain $s_0(M, A) < 0$ by Proposition 2.3.1.

We want to determine $\overline{\text{IV}_\varrho(M, A)}$. For doing so, let $\varrho > 0$ and $u_0 = (v, w) \in \text{IV}_\varrho(M, A)$. Then it follows that

$$S_\varrho(\delta_0 M_0 u_0) - \chi_{\mathbb{R}_{\geq 0}} u_0 \in H_\varrho^1(\mathbb{R}; H).$$

We set

$$\begin{pmatrix} \vartheta \\ q \end{pmatrix} := S_\varrho(\delta_0 M_0 u_0) = \left(\overline{\partial_{0,\varrho} \begin{pmatrix} \eta & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & k^{-1} \end{pmatrix} + \begin{pmatrix} 0 & -C^* \\ C & 0 \end{pmatrix}} \right)^{-1} \delta_0 \begin{pmatrix} \eta v \\ 0 \end{pmatrix},$$

which by definition needs to belong to $H_\varrho(\mathbb{R}; H)$. The equations read

$$\begin{aligned} \partial_{0,\varrho} \eta \vartheta - C^* q &= \delta_0 \eta v, \\ k^{-1} q + C \vartheta &= 0, \end{aligned}$$

and thus, the second equation yields $\vartheta \in D(C)$. Furthermore, since $q(t) \rightarrow w$ as $t \rightarrow 0+$, we derive that $kC\vartheta(t) = -q(t) \rightarrow -w$ as $t \rightarrow 0+$. Since also $\vartheta(t) \rightarrow v$ as $t \rightarrow 0+$, we infer from the closedness of C that $v \in D(C)$ and $w = -kCv$. Thus,

$$\text{IV}_\varrho(M, A) \subseteq \{(v, -kCv); v \in D(C)\}.$$

To find an appropriate subset of $\text{IV}_\varrho(M, A)$ we apply Proposition 3.4.1. Let $u_0 = (v, w) \in D(A)$. Then $(M_1 + A)u_0 \in R(M_0)$ yields

$$k^{-1}w + Cv = 0$$

and hence, $w = -kCv$. Since $w \in D(C^*)$ we infer $v \in D(C^*kC)$. Moreover, since we have that $\iota_0(\iota_1^* M_0 \iota_0)^{-1} \iota_1^*(M_1 + A)u_0 \in D(A)$, we derive

$$\eta^{-1}C^*w \in D(C)$$

and thus, $v \in D(C\eta^{-1}C^*kC)$. Thus, we have that

$$\{(v, -kCv); v \in D(C\eta^{-1}C^*kC)\} \subseteq \text{IV}_\varrho(M, A).$$

Both inclusions yield

$$\overline{\text{IV}_\varrho(M, A)} = \{(v, -kCv); v \in D(C)\} =: G.$$

If we show that (3.12) is satisfied, the semigroup $T_{(M,A)}^{(1)}(\cdot, 0)$ extends to a C_0 -semigroup on G . Let $v \in D(C)$, $\lambda > 0$ and define

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix} := \left(\begin{pmatrix} \lambda \eta & -C^* \\ C & k^{-1} \end{pmatrix}^{-1} \begin{pmatrix} \eta & 0 \\ 0 & 0 \end{pmatrix} \right)^n \begin{pmatrix} v \\ -kCv \end{pmatrix}$$

for $n \in \mathbb{N}$. Then,

$$u_n = \left((\lambda \eta + C^*kC)^{-1} \eta \right)^n v$$

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$$v_n = -kC \left((\lambda\eta + C^*kC)^{-1} \eta \right)^n v.$$

First, we estimate u_n in terms of v . For doing so, we compute

$$(\lambda\eta + C^*kC)^{-1} = \sqrt{\eta^{-1}} \left(\lambda + \sqrt{\eta^{-1}} C^* k C \sqrt{\eta^{-1}} \right)^{-1} \sqrt{\eta^{-1}}$$

and consequently

$$\begin{aligned} \left((\lambda\eta + C^*kC)^{-1} \eta \right)^n &= \left(\sqrt{\eta^{-1}} \left(\lambda + \sqrt{\eta^{-1}} C^* k C \sqrt{\eta^{-1}} \right)^{-1} \sqrt{\eta} \right)^n \\ &= \sqrt{\eta^{-1}} \left(\lambda + \sqrt{\eta^{-1}} C^* k C \sqrt{\eta^{-1}} \right)^{-n} \sqrt{\eta}. \end{aligned}$$

Since $\sqrt{\eta^{-1}} C^* k C \sqrt{\eta^{-1}}$ is a selfadjoint accretive operator on H_0 , we get that

$$|u_n|_{H_0} \leq \frac{\|\sqrt{\eta}\| \sqrt{c_0^{-1}}}{\lambda^n} |v|_{H_0}.$$

Moreover, by the above computations, we have that

$$\begin{aligned} -kC \left((\lambda\eta + C^*kC)^{-1} \eta \right)^n &= -kC \sqrt{\eta^{-1}} \left(\lambda + \sqrt{\eta^{-1}} C^* k C \sqrt{\eta^{-1}} \right)^{-n} \sqrt{\eta} \\ &\supseteq \sqrt{k} (\lambda + \sqrt{k} C \eta^{-1} C^* \sqrt{k})^{-n} (-\sqrt{k} C) \end{aligned}$$

and thus,

$$v_n = \sqrt{k} \left(\lambda + \sqrt{k} C \eta^{-1} C^* \sqrt{k} \right)^{-n} (-\sqrt{k} C v).$$

Since $\sqrt{k} C \eta^{-1} C^* \sqrt{k}$ is a selfadjoint accretive operator, we infer

$$|v_n|_{H_1} \leq \frac{\|\sqrt{k}\|}{\lambda^n} |\sqrt{k} C v|_{H_1} \leq \frac{\|\sqrt{k}\| \sqrt{c_1^{-1}}}{\lambda^n} |k C v|_{H_1}.$$

Summarizing, we have shown that

$$\left| \left(\begin{pmatrix} \lambda\eta & -C^* \\ C & k^{-1} \end{pmatrix}^{-1} \begin{pmatrix} \eta & 0 \\ 0 & 0 \end{pmatrix} \right)^n \begin{pmatrix} v \\ -k C v \end{pmatrix} \right|_H \leq \frac{M}{\lambda^n} \left| \begin{pmatrix} v \\ -k C v \end{pmatrix} \right|_H$$

for some $M \geq 1$, which is (3.12) according to Remark 3.4.6 (a). Moreover $\omega(T_{(M,A)}^{(1)}(\cdot, 0)) \leq s_0(M, A)$ by Proposition 3.4.7.

Remark 3.4.10. If we choose $H_0 = L_2(\Omega)$ and $H_1 = R(\text{grad}_0)$ for some bounded domain $\Omega \subseteq \mathbb{R}^3$ and $C := \iota_{R(\text{grad}_0)}^* \text{grad}_0$, we end up with the heat equation. Thus, we can associate a C_0 -semigroup on the Hilbert space $G = \{(v, -k \text{grad}_0 v) ; v \in D(\text{grad}_0)\} \subseteq L_2(\Omega) \oplus R(\text{grad}_0)$, which is exponentially stable, since C is boundedly invertible due to the Poincaré inequality. We emphasize that the C_0 -semigroup is not the classical one, since we require the continuity with respect to time for both unknowns, the temperature and the heat flux. This is why we need the Hilbert space G instead of the usual space $L_2(\Omega)$.

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3.4.2. Partial differential equations with finite delay

We generalize the setting of the previous subsection and consider now material laws of the form

$$M(z) := M_0 + z^{-1}(M_1 + \sum_{i=2}^n M_i e^{-zh_i}) \quad (z \in \mathbb{C} \setminus \{0\}),$$

where $M_0, M_1, M_i \in L(H)$ and $h_i > 0, i \in \{2, \dots, n\}$. Then, M satisfies (REG) with $b(M) = 0$ and for $\varrho > 0, x \in H$ we get

$$M(\partial_{0,\varrho})\chi_{\mathbb{R}_{\geq 0}}x = \left(t \mapsto \chi_{\mathbb{R}_{\geq 0}}(t)M_0x + t\chi_{\mathbb{R}_{\geq 0}}(t)M_1x + \sum_{i=2}^n (t - h_i)\chi_{\mathbb{R}_{\geq h_i}}(t)M_ix \right)$$

which yields

$$(M(\partial_{0,\varrho})\chi_{\mathbb{R}_{\geq 0}}x)(0+) = M_0x.$$

Moreover, we have that

$$\begin{aligned} K_\varrho g &= P_0 \partial_{0,\varrho} M(\partial_{0,\varrho})g \\ &= \chi_{\mathbb{R}_{\geq 0}}(m) \sum_{i=2}^n M_i \tau_{-h_i} g \end{aligned}$$

for each $g \in \chi_{\mathbb{R}_{\leq 0}}(m) [H_\varrho^1(\mathbb{R}; H)]$. Thus, a function $g \in \chi_{\mathbb{R}_{\leq 0}}(m) [H_\varrho^1(\mathbb{R}; H)]$ for some $\varrho > \max\{s_0(M, A), 0\}$ belongs to the space of admissible histories $\text{His}_\varrho(M, A)$ if and only if

$$S_\varrho \left(\delta_0 M_0 g(0-) - \chi_{\mathbb{R}_{\geq 0}}(m) \sum_{i=2}^n M_i \tau_{-h_i} g \right) - \chi_{\mathbb{R}_{\geq 0}} g(0-) \in H_\varrho^1(\mathbb{R}; H).$$

Similar to the previous section, we describe a subset of $\text{His}_\varrho(M, A)$.

Proposition 3.4.11. *Let $R(M_0)$ be closed and denote by $\iota_0 : N(M_0)^\perp \rightarrow H$ and $\iota_1 : R(M_0) \rightarrow H$ the canonical embeddings of $N(M_0)^\perp$ and $R(M_0)$, respectively. Then, $\iota_1^* M_0 \iota_0 : N(M_0)^\perp \rightarrow R(M_0)$ is boundedly invertible and each function $g \in \chi_{\mathbb{R}_{\leq 0}}(m) [H_\varrho^1(\mathbb{R}; H)]$, $\varrho > 0$ satisfying*

- $g(0-) \in D(A)$,
- $(M_1 + A)g(0-) \in R(M_0)$,
- $\iota_0(\iota_1^* M_0 \iota_0)^{-1} \iota_1^*(M_1 + A)g(0-) \in D(A)$,
- $M_i g(-h_i) = 0$ for each $i \in \{2, \dots, n\}$,

belongs to $\text{His}_\varrho(M, A)$.

Proof. The bounded invertibility of $\iota_1^* M_0 \iota_0$ follows by the closed graph theorem. Let now $g \in \chi_{\mathbb{R}_{\leq 0}}(m) [H_\varrho^1(\mathbb{R}; H)]$, $\varrho > 0$ satisfying the conditions. We need to prove that

$$S_\varrho \left(\delta_0 M_0 g(0-) - \chi_{\mathbb{R}_{\geq 0}}(m) \sum_{i=2}^n M_i \tau_{-h_i} g \right) - \chi_{\mathbb{R}_{\geq 0}} g(0-) \in H_\varrho^1(\mathbb{R}; H).$$

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Since $g(0-) \in D(A)$ we get that

$$\begin{aligned} & S_\varrho \left(\delta_0 M_0 g(0-) - \chi_{\mathbb{R}_{\geq 0}}(\mathfrak{m}) \sum_{i=2}^n M_i \tau_{-h_i} g \right) - \chi_{\mathbb{R}_{\geq 0}} g(0-) \\ &= -S_\varrho \left(\chi_{\mathbb{R}_{\geq 0}}(\mathfrak{m}) \sum_{i=2}^n M_i \tau_{-h_i} g + \chi_{\mathbb{R}_{\geq 0}}(M_1 + A)g(0-) + \sum_{i=2}^n M_i \tau_{-h_i} \chi_{\mathbb{R}_{\geq 0}} g(0-) \right). \end{aligned}$$

Note that by assumption there is $x \in D(A)$ such that $M_0 x = (M_1 + A)g(0-)$. Moreover, $\text{spt } \tau_{-h_i} \chi_{\mathbb{R}_{\geq 0}} g(0-) \subseteq \mathbb{R}_{\geq h_i}$ and thus,

$$\begin{aligned} & S_\varrho \left(\delta_0 M_0 g(0-) - \chi_{\mathbb{R}_{\geq 0}}(\mathfrak{m}) \sum_{i=2}^n M_i \tau_{-h_i} g \right) - \chi_{\mathbb{R}_{\geq 0}} g(0-) \\ &= -S_\varrho(\chi_{\mathbb{R}_{\geq 0}} M_0 x) - S_\varrho \left(\chi_{\mathbb{R}_{\geq 0}}(\mathfrak{m}) \sum_{i=2}^n M_i \tau_{-h_i} (g + \chi_{\mathbb{R}_{\geq 0}} g(0-)) \right). \end{aligned}$$

Note that $g + \chi_{\mathbb{R}_{\geq 0}} g(0-) \in H_\varrho^1(\mathbb{R}; H)$ and thus, $M_i \tau_{-h_i} (g + \chi_{\mathbb{R}_{\geq 0}} g(0-)) \in H_\varrho^1(\mathbb{R}; H)$. Furthermore, by hypothesis we have that

$$(M_i \tau_{-h_i} (g + \chi_{\mathbb{R}_{\geq 0}} g(0-))) (0) = M_i g(-h_i) = 0$$

and hence,

$$\chi_{\mathbb{R}_{\geq 0}}(\mathfrak{m}) \sum_{i=2}^n M_i \tau_{-h_i} (g + \chi_{\mathbb{R}_{\geq 0}} g(0-)) \in H_\varrho^1(\mathbb{R}; H).$$

Thus, it suffices to check that

$$S_\varrho(\chi_{\mathbb{R}_{\geq 0}} M_0 x) \in H_\varrho^1(\mathbb{R}; H).$$

This, however, follows by

$$\partial_{0,\varrho} S_\varrho(\chi_{\mathbb{R}_{\geq 0}} M_0 x) = \chi_{\mathbb{R}_{\geq 0}} x - S_\varrho \left(\chi_{\mathbb{R}_{\geq 0}}(M_1 + A)x + \sum_{i=2}^n M_i \tau_{-h_i} \chi_{\mathbb{R}_{\geq 0}} x \right) \in H_\varrho(\mathbb{R}; H),$$

where we have used $x \in D(A)$. □

The main difficulty lies in the verification of (HY) in order to obtain a C_0 -semigroup on X_ϱ^μ for some $\varrho > s_0(M, A)$ and $\mu \leq \varrho$. However, if (HY) is satisfied, Conditions (GP-1) and (GP-2) follow and hence, we can estimate the growth-type of the semigroup by μ .

Proposition 3.4.12. *Let $\varrho > \max\{s_0(M, A), 0\}$ and $s_0(M, A) < \mu \leq \varrho$. Assume that $T_{(M,A)}$ extends to a C_0 -semigroup on X_ϱ^μ . Then $\omega(T_{(M,A)}) \leq \mu$.*

Proof. By Theorem 3.3.7 it suffices to prove (GP-1) and (GP-2). We observe that for $z \in$

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$\mathbb{C}_{\operatorname{Re} > \mu}$ and $\lambda > 0$ we have

$$\begin{aligned} \|\Phi_\lambda(z)\| &= \|zM_0 + M_1 + \sum_{i=2}^n M_i e^{-zh_i} - \left((z + \lambda)M_0 + M_1 + \sum_{i=2}^n M_i e^{-(z+\lambda)h_i}\right)\| \\ &= \left\| -\lambda M_0 + \sum_{i=2}^n M_i e^{-zh_i} (1 - e^{-\lambda h_i}) \right\| \\ &\leq \lambda \|M_0\| + \sum_{i=2}^n \|M_i\| e^{-\mu h_i} \end{aligned}$$

which shows (GP-1). Moreover, for $g \in \chi_{\mathbb{R}_{\leq 0}}(\mathfrak{m}) [H_\varrho^1(\mathbb{R}; H)]$ we estimate

$$\begin{aligned} \left(\int_0^\infty |K_\varrho g(t)|_H^2 e^{-2\mu t} dt \right)^{\frac{1}{2}} &= \left(\int_0^\infty \left| \sum_{i=2}^n M_i g(t - h_i) \right|_H^2 e^{-2\mu t} dt \right)^{\frac{1}{2}} \\ &\leq \sum_{i=2}^n \|M_i\| e^{-\mu h_i} \left(\int_{-h_i}^0 |g(s)|_H^2 e^{-2\mu s} ds \right)^{\frac{1}{2}} \\ &\leq \sum_{i=2}^n \|M_i\|^2 e^{-\mu h_i} |g|_{H_\varrho(\mathbb{R}_{\leq 0}; H)}, \end{aligned}$$

which proves (GP-2). Hence, the assertion follows. \square

We now discuss a more concrete example.

On a Cauchy-type problem with finite delay

Let $A : D(A) \subseteq H \rightarrow H$ such that $-A$ is the generator of a C_0 -semigroup T on H and $M_i \in L(H)$, $i \in \{2, \dots, n\}$. We consider the following evolutionary problem

$$\left(\partial_{0, \varrho} + \sum_{i=2}^n M_i \tau_{-h_i} + A \right) u = f,$$

where $h_i > 0$ for each $i \in \{2, \dots, n\}$. This problem is well-posed, since for $z \in \mathbb{C}_{\operatorname{Re} > \varrho}$ with $\varrho > \max\{0, \omega(T)\}$ sufficiently large, we have that

$$z + \sum_{i=2}^n M_i e^{-zh_i} + A = (z + A) \left(1 + (z + A)^{-1} \sum_{i=2}^n M_i e^{-zh_i} \right),$$

and

$$\|(z + A)^{-1} \sum_{i=2}^n M_i e^{-zh_i}\| \leq \frac{M \sum_{i=2}^n \|M_i\| e^{-\operatorname{Re} z h_i}}{\operatorname{Re} z - \omega(T)} \leq \frac{M \sum_{i=2}^n \|M_i\|}{\varrho - \omega(T)} e^{-\varrho h},$$

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for some $M \geq 1$, where $h := \min_{2 \leq i \leq n} h_i$. Thus, choosing ϱ large enough such that

$$\frac{M \sum_{i=2}^n \|M_i\|}{\varrho - \omega(T)} e^{-\varrho h} < 1, \quad (3.13)$$

we derive the well-posedness by employing the Neumann-series.

Lemma 3.4.13. *For $\varrho > \max\{s_0(M, A), 0\}$ we have that*

$$X_\varrho^\mu = H \times H_\mu(\mathbb{R}_{\leq 0}; H)$$

for each $\mu \leq \varrho$.

Proof. We have to prove that $D_\varrho = \{(g(0-), g); g \in \text{His}_\varrho(M, A)\}$ is dense in $H \times H_\mu(\mathbb{R}_{\leq 0}; H)$. Let $v \in H$ and $f \in H_\mu(\mathbb{R}_{\leq 0}; H)$. For $\varepsilon > 0$ we take $x \in D(A^2)$ such that

$$|x - v|_H < \varepsilon$$

and $\varphi \in C_c^\infty(\mathbb{R}_{< 0}; H)$ such that

$$|\varphi - f|_{H_\mu(\mathbb{R}_{\leq 0}; H)} < \varepsilon.$$

Moreover, we choose $\psi \in C_c^\infty(\mathbb{R}; H)$ with $|\psi|_{H_\mu(\mathbb{R}; H)} < \varepsilon$ such that $\psi(0) = x$ and $\psi(-h_i) = -\varphi(-h_i)$ for each $i \in \{2, \dots, n\}$. We define

$$g := \chi_{\mathbb{R}_{\leq 0}}(\mathfrak{m})(\varphi + \psi) \in \chi_{\mathbb{R}_{\leq 0}}(\mathfrak{m})[H_\varrho^1(\mathbb{R}; H)]$$

and show that $g \in \text{His}_\varrho(M, A)$. For doing so, we apply Proposition 3.4.11 and show that g satisfies the four conditions stated there. We have that

- $g(0-) = \psi(0) = x \in D(A)$,
- $(M_1 + A)g(0-) = Ax \in H = R(M_0)$,
- $\iota_0(\iota_1^* M_0 \iota_0)^{-1} \iota_1^*(M_1 + A)g(0-) = Ax \in D(A)$,
- $M_i g(-h_i) = M_i(\varphi(-h_i) + \psi(-h_i)) = 0$ for $i \in \{2, \dots, n\}$.

Hence, we indeed have $g \in \text{His}_\varrho(M, A)$. Moreover,

$$\begin{aligned} |g(0-) - v|_H &= |x - v|_H < \varepsilon, \\ |g - f|_{H_\mu(\mathbb{R}_{\leq 0}; H)} &\leq |\varphi - f|_{H_\mu(\mathbb{R}_{\leq 0}; H)} + |\psi|_{H_\mu(\mathbb{R}; H)} < 2\varepsilon, \end{aligned}$$

which proves the density. □

Proposition 3.4.14. *Let $\varrho > \max\{s_0(M, A), 0\}$ such that (3.13) is satisfied. Then for each $\mu \leq \varrho$, $T_{(M, A)}$ can be extended to a C_0 -semigroup on X_ϱ^μ .*

Proof. By Proposition 3.3.3 and Lemma 3.3.5 it suffices to prove that there is some $\omega \in \mathbb{R}$ such that

$$T_{(M, A)}^{(1)} : D_\varrho \subseteq X_\varrho^\mu \rightarrow C_\omega(\mathbb{R}_{\geq 0}; H)$$

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is bounded. We observe that for $\varrho > \omega(T)$

$$(\partial_{0,\varrho} + A)^{-1} : H_\varrho(\mathbb{R}_{\geq 0}; H) \rightarrow C_\varrho(\mathbb{R}_{\geq 0}; H) \quad (3.14)$$

is bounded. Indeed, for $f \in C_c^\infty(\mathbb{R}_{\geq 0}; H)$ we estimate

$$\begin{aligned} \left| \left((\partial_{0,\varrho} + A)^{-1} f \right) (t) \right|_H e^{-\varrho t} &= \left| \int_0^t T(t-s) f(s) \, ds \right|_H e^{-\varrho t} \\ &\leq M e^{(\omega(T)-\varrho)t} \int_0^t |f(s)|_H e^{-\omega(T)s} \, ds \\ &\leq M e^{(\omega(T)-\varrho)t} \sqrt{\frac{e^{2(\varrho-\omega(T))t} - 1}{2(\varrho - \omega(T))}} |f|_{H_\varrho(\mathbb{R}_{\geq 0}; H)} \\ &\leq M \frac{1}{\sqrt{2(\varrho - \omega(T))}} |f|_{H_\varrho(\mathbb{R}_{\geq 0}; H)}, \end{aligned}$$

which shows the claim. Let now $\varrho > \omega(T)$ satisfying (3.13) and $\mu \leq \varrho$. For $(x, g) \in X_\varrho^\mu$ we set

$$\begin{aligned} u &:= \left(\partial_{0,\varrho} + \sum_{i=2}^n M_i \tau_{h_i} + A \right)^{-1} \left(\delta_0 x - \chi_{\mathbb{R}_{\geq 0}}(\mathbf{m}) \sum_{i=2}^n M_i \tau_{h_i} g \right) \\ &= \left(1 + (\partial_{0,\varrho} + A)^{-1} \sum_{i=2}^n M_i \tau_{h_i} \right)^{-1} (\partial_{0,\varrho} + A)^{-1} \delta_0 x + \\ &\quad - \left(\partial_{0,\varrho} + \sum_{i=2}^n M_i \tau_{h_i} + A \right)^{-1} \chi_{\mathbb{R}_{\geq 0}}(\mathbf{m}) \sum_{i=2}^n M_i \tau_{h_i} g. \end{aligned}$$

Since $(\partial_{0,\varrho} + A)^{-1} \delta_0 x \in H_\varrho(\mathbb{R}_{\geq 0}; H)$ and

$$\chi_{\mathbb{R}_{\geq 0}}(\mathbf{m}) \tau_{h_i} g = \chi_{[0, h_i]}(\mathbf{m}) \tau_{h_i} g \in H_\mu([0, h_i]; H) \hookrightarrow H_\varrho(\mathbb{R}; H),$$

we derive that $u \in H_\varrho(\mathbb{R}_{\geq 0}; H)$ and

$$|u|_{H_\varrho(\mathbb{R}; H)} \leq C |(x, g)|_{X_\varrho^\mu}$$

for some $C > 0$. We thus have shown that

$$T_{(M,A)}^{(1)} : X_\varrho^\mu \rightarrow H_\varrho(\mathbb{R}_{\geq 0}; H)$$

is bounded. Moreover, we have that

$$\left(\partial_{0,\varrho} + \sum_{i=2}^n M_i \tau_{h_i} + A \right) u = \delta_0 x - \chi_{\mathbb{R}_{\geq 0}}(\mathbf{m}) \sum_{i=2}^n M_i \tau_{h_i} g,$$

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which gives

$$u = (\partial_{0,\varrho} + A)^{-1} \delta_0 x - (\partial_{0,\varrho} + A)^{-1} \left(\chi_{\mathbb{R}_{\geq 0}}(\mathfrak{m}) \sum_{i=2}^n M_i \tau_{h_i} g - \sum_{i=2}^n M_i \tau_{h_i} u \right).$$

Since $-A$ generates a C_0 -semigroup, the first term on the right-hand side belongs to $C_\varrho(\mathbb{R}_{\geq 0}; H)$. Moreover, employing (3.14) we infer that also the second term lies in $C_\varrho(\mathbb{R}_{\geq 0}; H)$ and hence, so does u . Hence, we have shown

$$T_{(M,A)}^{(1)} [X_\varrho^\mu] \subseteq C_\varrho(\mathbb{R}_{\geq 0}; H).$$

Since $T_{(M,A)}^{(1)} \in L(X_\varrho^\mu; H_\varrho(\mathbb{R}_{\geq 0}; H)) \subseteq L(X_\varrho^\mu; H_{\varrho+1}(\mathbb{R}_{\geq 0}; H))$ and $C_\varrho(\mathbb{R}_{\geq 0}; H) \hookrightarrow H_{\varrho+1}(\mathbb{R}_{\geq 0}; H)$, we infer that $T_{(M,A)}^{(1)} : X_\varrho^\mu \rightarrow C_\varrho(\mathbb{R}_{\geq 0}; H)$ is bounded by the closed graph theorem. \square

Corollary 3.4.15. *Assume that $A - c$ is m -accretive for some $c > 0$. Moreover, assume that M_i is selfadjoint and non-negative for each $i \in \{2, \dots, n\}$. Then $s_0(M, A) \leq -c$ and consequently, the semigroup $T_{(M,A)}$ on X_ϱ^μ for $s_0(M, A) < \mu < 0$ is exponentially stable.*

Proof. For $x \in H$ and $z \in \mathbb{C}$ we have

$$\operatorname{Re} \left\langle \left(z + \sum_{i=2}^n M_i e^{-zh_i} + A \right) x | x \right\rangle \geq (\operatorname{Re} z + c) |x|_H^2$$

and hence, $s_0(M, A) \leq -c$. The second assertion follows from Proposition 3.4.12. \square

3.4.3. Integro-differential equations

A further class we want to inspect with the methods provided in this chapter is a particular class of integro-differential equations. For simplicity, we restrict ourselves to integro-differential equations of parabolic-type.

Let H be a Hilbert space and $A : D(A) \subseteq H \rightarrow H$ an m -accretive operator. Moreover, let $k : \mathbb{R}_{\geq 0} \rightarrow L(H)$ be a given kernel satisfying $k \in L_{1,\mu}(\mathbb{R}_{\geq 0}; L(H))$ with $|k|_{L_{1,\mu}} < 1$ for some $\mu \in \mathbb{R}$ (recall the definitions given in 1.3.3). We consider an integro-differential equation of the form

$$\left(\partial_{0,\varrho} (1 - k*)^{-1} + A \right) u = f, \quad (3.15)$$

for $\varrho \geq \mu$. This is an evolutionary equation with a material law given by

$$M(z) := (1 - \sqrt{2\pi} \widehat{k}(z))^{-1} \quad (z \in \mathbb{C}_{\operatorname{Re} > \mu}).$$

To avoid technicalities we assume that H is separable. We first check that M satisfies (REG). Clearly, we have $b(M) \leq \max\{0, \mu\}$ due to the Neumann series. In order to prove that

$$((1 - k*)^{-1} \chi_{\mathbb{R}_{\geq 0}} x) (0+)$$

exists for each $x \in H$, we need the following lemma.

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Lemma 3.4.16. *For $\varrho \in \mathbb{R}$ consider the space*

$$CH_\varrho(\mathbb{R}; H) := C_\varrho(\mathbb{R}; H) \cap H_\varrho(\mathbb{R}; H)$$

equipped with the norm

$$|f|_{CH_\varrho} := |f|_{\varrho, \infty} + |f|_{H_\varrho(\mathbb{R}; H)} \quad (f \in CH_\varrho(\mathbb{R}; H)).$$

Then $CH_\varrho(\mathbb{R}; H)$ is a Banach space and for $\ell \in L_{1, \varrho}(\mathbb{R}_{\geq 0}; L(H))$ the operator

$$\ell * : CH_\varrho(\mathbb{R}; H) \rightarrow CH_\varrho(\mathbb{R}; H)$$

is well-defined and bounded with

$$\|\ell * \| \leq \|\ell\|_{L_{1, \varrho}}.$$

Proof. If $(f_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $CH_\varrho(\mathbb{R}; H)$ it converges to f and \tilde{f} in $C_\varrho(\mathbb{R}; H)$ and $H_\varrho(\mathbb{R}; H)$, respectively. Then there is a subsequence $(f_{n_k})_{k \in \mathbb{N}}$ such that $f_{n_k}(t) \rightarrow \tilde{f}(t)$ for almost every $t \in \mathbb{R}$. Since $f_n(t) \rightarrow f(t)$ for each $t \in \mathbb{R}$ we infer $f(t) = \tilde{f}(t)$ for almost every $t \in \mathbb{R}$, which proves that $f \in CH_\varrho(\mathbb{R}; H)$ is the limit of $(f_n)_{n \in \mathbb{N}}$ in $CH_\varrho(\mathbb{R}; H)$. Let now $\ell \in L_{1, \varrho}(\mathbb{R}_{\geq 0}; L(H))$. We first show that $\ell * f$ is continuous for $f \in CH_\varrho(\mathbb{R}; H)$. Recall that by Lemma 1.3.6

$$(\ell * f)(t) = \int_0^\infty \ell(s) f(t-s) \, ds \quad (t \in \mathbb{R} \text{ a.e.}).$$

Let $t, t' \in \mathbb{R}$ with $|t - t'| \leq 1$. Then we have that

$$\left| \int_0^\infty \ell(s) (f(t-s) - f(t'-s)) \, ds \right|_H \leq \int_0^\infty \|\ell(s)\| |f(t-s) - f(t'-s)|_H \, ds.$$

By continuity of f , we have $f(t'-s) \rightarrow f(t-s)$ for each $s \in \mathbb{R}$ as $t' \rightarrow t$. Moreover, we have that

$$\begin{aligned} \|\ell(s)\| |f(t-s) - f(t'-s)|_H &\leq \|\ell(s)\| e^{-\varrho s} \left(|f(t-s)|_H e^{-\varrho(t-s)} + |f(t'-s)|_H e^{-\varrho(t-s)} \right) e^{\varrho t} \\ &\leq \|\ell(s)\| e^{-\varrho s} \left(1 + e^{-\varrho(t-t')} \right) |f|_{\varrho, \infty} e^{\varrho t} \\ &\leq \|\ell(s)\| e^{-\varrho s} \left(1 + e^{|\varrho|} \right) |f|_{\varrho, \infty} e^{\varrho t} =: g(s) \end{aligned}$$

for each $s \in \mathbb{R}$ and since $g \in L_1(\mathbb{R}_{\geq 0})$ we obtain

$$\int_0^\infty \|\ell(s)\| |f(t-s) - f(t'-s)|_H \, ds \rightarrow 0 \quad (t' \rightarrow t)$$

by dominated convergence. Hence $(t \mapsto \int_0^\infty \ell(s) f(t-s) \, ds)$ is a continuous representer of

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$\ell * f$. Moreover,

$$\left| \int_0^\infty \ell(s) f(t-s) \, ds \right|_H e^{-\varrho t} \leq \int_0^\infty \|\ell(s)\| e^{-\varrho s} |f(t-s)|_H e^{-\varrho(t-s)} \, ds \leq |\ell|_{L_{1,\varrho}} |f|_{\varrho,\infty}$$

for each $t \in \mathbb{R}$. Together with Lemma 1.3.5 this shows that $\ell * : CH_\varrho(\mathbb{R}; H) \rightarrow CH_\varrho(\mathbb{R}; H)$ is bounded with $\|\ell * \| \leq |\ell|_{L_{1,\varrho}}$. \square

Proposition 3.4.17. *For each $x \in H$ we have that*

$$((1 - k*)^{-1} \chi_{\mathbb{R}_{\geq 0}} x) (0+) = x.$$

Proof. Let $\varrho > \max\{0, \mu\}$ and $x \in H$. We first prove that $k * \chi_{\mathbb{R}_{\geq 0}} x \in CH_\varrho(\mathbb{R}; H)$. For $t, t' \in \mathbb{R}$ we have that

$$\int_0^\infty k(s) \chi_{\mathbb{R}_{\geq 0}}(t' - s) x \, ds = \int_0^\infty \chi_{[0,t']}(s) k(s) x \, ds \rightarrow \int_0^\infty \chi_{[0,t]}(s) k(s) x \, ds = \int_0^\infty k(s) \chi_{\mathbb{R}_{\geq 0}}(t - s) x \, ds$$

as $t' \rightarrow t$ by dominated convergence. Hence, by Lemma 1.3.6 ($t \mapsto \int_0^\infty k(s) \chi_{\mathbb{R}_{\geq 0}}(t - s) x \, ds$) is a continuous representer of $k * \chi_{\mathbb{R}_{\geq 0}} x$. Since

$$\left| \int_0^\infty k(s) \chi_{\mathbb{R}_{\geq 0}}(t - s) x \, ds \right|_H e^{-\varrho t} \leq \int_0^\infty \|k(s)\| e^{-\varrho s} \, ds |x|_H \quad (t \in \mathbb{R}),$$

we obtain $f := k * \chi_{\mathbb{R}_{\geq 0}} x \in CH_\varrho(\mathbb{R}; H)$. By the Neumann series we have

$$(1 - k*)^{-1} \chi_{\mathbb{R}_{\geq 0}} x = \chi_{\mathbb{R}_{\geq 0}} x + \sum_{n=0}^\infty (k*)^n f.$$

We note that the series also converges in $CH_\varrho(\mathbb{R}; H)$, since by Lemma 3.4.16

$$|(k*)^n f|_{CH_\varrho} \leq |k|_{L_{1,\varrho}}^n |f|_{CH_\varrho}$$

for each $n \in \mathbb{N}$ and $CH_\varrho(\mathbb{R}; H)$ is complete. Thus, $\sum_{n=0}^\infty (k*)^n f \in CH_\varrho(\mathbb{R}; H)$ and since $\text{spt } f \subseteq \mathbb{R}_{\geq 0}$ and $k*$ is causal, we have that $\text{spt } \sum_{n=0}^\infty (k*)^n f \subseteq \mathbb{R}_{\geq 0}$ and hence, using the continuity

$$\left(\sum_{n=0}^\infty (k*)^n f \right) (0+) = \left(\sum_{n=0}^\infty (k*)^n f \right) (0-) = 0.$$

Hence, we get that

$$((1 - k*)^{-1} \chi_{\mathbb{R}_{\geq 0}} x) (0+) = x + \left(\sum_{n=0}^\infty (k*)^n f \right) (0+) = x. \quad \square$$

Our next goal is to find a suitable subset of $\text{His}_\varrho(M, A)$. For doing so, we need to impose more regularity for the kernel k . More precisely, we assume the following: There exists a kernel

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$k' \in L_{1,\tilde{\mu}}(\mathbb{R}_{\geq 0}; L(H))$ for some $\tilde{\mu} \in \mathbb{R}$ such that for each $x, y \in H, t \in \mathbb{R}_{\geq 0}$ we have

$$\langle x | (k(t) - k(0))y \rangle_H = \int_0^t \langle x | k'(s)y \rangle_H \, ds. \quad (3.16)$$

Lemma 3.4.18. *Assume that k satisfies (3.16) and let $u \in H_\varrho(\mathbb{R}; H)$ for some $\varrho \geq \max\{\mu, \tilde{\mu}\}$. Then $k * u \in H_\varrho^1(\mathbb{R}; H)$ and $\partial_{0,\varrho}(k * u) = k' * u + k(0)u$. Hence,*

$$k* : H_\varrho(\mathbb{R}; H) \rightarrow H_\varrho^1(\mathbb{R}; H)$$

is a bounded linear operator with norm less than or equal to $|k'|_{L_{1,\varrho}} + \|k(0)\|$ and

$$z\widehat{k}(z) = \widehat{k}'(z) + \frac{1}{\sqrt{2\pi}}k(0) \quad (z \in \mathbb{C}_{\operatorname{Re} \geq \max\{\mu, \tilde{\mu}\}}).$$

Proof. Let $\varphi \in C_c^\infty(\mathbb{R}; H)$ and compute

$$\begin{aligned} & \langle k * u | \partial_{0,\varrho}^* \varphi \rangle_{H_\varrho(\mathbb{R}; H)} \\ &= - \int_{\mathbb{R}} \left\langle \int_{-\infty}^t k(t-s)u(s) \, ds \middle| (\varphi e^{-2\varrho \cdot})'(t) \right\rangle_H \, dt \\ &= - \int_{\mathbb{R}} \int_s^\infty \langle k(t-s)u(s) | (\varphi e^{-2\varrho \cdot})'(t) \rangle_H \, dt \, ds \\ &= - \int_{\mathbb{R}} \int_s^\infty \left\langle \int_0^{t-s} k'(r)u(s) \, dr | (\varphi e^{-2\varrho \cdot})'(t) \right\rangle_H \, dt \, ds - \int_{\mathbb{R}} \langle k(0)u(s) | \int_s^\infty (\varphi e^{-2\varrho \cdot})'(t) \, dt \rangle_H \, ds \\ &= \int_{\mathbb{R}} \int_s^\infty \langle k'(t-s)u(s) | \varphi(t) e^{-2\varrho t} \rangle_H \, dt \, ds + \int_{\mathbb{R}} \langle k(0)u(s) | \varphi(s) \rangle_H e^{-2\varrho s} \, ds \\ &= \langle k' * u + k(0)u | \varphi \rangle_{H_\varrho(\mathbb{R}; H)}. \end{aligned}$$

This yields $\partial_{0,\varrho}(k * u) = k' * u + k(0)u$. The boundedness of $k* : H_\varrho(\mathbb{R}; H) \rightarrow H_\varrho^1(\mathbb{R}; H)$ with the asserted norm bound follows from Lemma 1.3.5, while the last assertion is a consequence of Lemma 1.3.8. \square

Proposition 3.4.19. *If k satisfies (3.16), then the evolutionary problem associated with A and $M(z) = (1 - \sqrt{2\pi}\widehat{k}(z))^{-1}$ is well-posed. Moreover, for $\varrho \geq \max\{\mu, \tilde{\mu}\}$ we have that*

$$\begin{aligned} K_\varrho : H_\varrho(\mathbb{R}_{\leq 0}; H) &\rightarrow H_{\mu \vee \tilde{\mu}}(\mathbb{R}_{\geq 0}; H) \\ g &\mapsto P_0 \partial_0 M(\partial_0)g \end{aligned}$$

is well-defined and bounded and, for $\mu \vee \tilde{\mu} \leq \nu \leq \varrho$ and $g \in H_\varrho(\mathbb{R}_{\leq 0}; H) \subseteq H_\nu(\mathbb{R}_{\leq 0}; H)$ we have that

$$K_\varrho g = K_\nu g.$$

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Proof. Since $|k|_{L_{1,\mu}} < 1$ we have that

$$M(z) = 1 + \sum_{n=1}^{\infty} \left(\sqrt{2\pi} \widehat{k}(z) \right)^n, \quad (z \in \mathbb{C}_{\text{Re} > \mu})$$

due to the Neumann series. From Lemma 3.4.18, we therefore infer that

$$zM(z) = z + \left(\sqrt{2\pi} \widehat{k}'(z) + k(0) \right) \sum_{n=0}^{\infty} \left(\sqrt{2\pi} \widehat{k}(z) \right)^n =: z + \widetilde{M}(z) \quad (z \in \mathbb{C}_{\text{Re} > \max\{\mu, \widetilde{\mu}\}}).$$

Noting that

$$\|\widetilde{M}(z)\| \leq (|k'|_{L_{1,\widetilde{\mu}}} + \|k(0)\|) \frac{1}{1 - |k|_{L_{1,\mu}}} =: \varrho_0$$

for each $z \in \mathbb{C}_{\text{Re} > \max\{\mu, \widetilde{\mu}\}}$ we obtain that

$$zM(z) + A = z + A + \widetilde{M}(z) = (z + A)(1 + (z + A)^{-1} \widetilde{M}(z))$$

is boundedly invertible for each $z \in \mathbb{C}_{\text{Re} > \max\{0, \varrho_0, \mu, \widetilde{\mu}\}}$ with

$$\|(zM(z) + A)^{-1}\| \leq \frac{1}{\text{Re } z - \varrho_0}.$$

This shows the well-posedness of the evolutionary problem associated with M and A .

Let now $g \in H_{\varrho}(\mathbb{R}_{\leq 0}; H)$ for some $\varrho \geq \max\{\mu, \widetilde{\mu}\}$. Then we have by Lemma 3.4.18

$$\begin{aligned} \partial_{0,\varrho} M(\partial_{0,\varrho})g &= \partial_{0,\varrho} g + \partial_{0,\varrho} \sum_{n=1}^{\infty} (k*)^n g \\ &= \partial_{0,\varrho} g + (k' * + k(0)) \sum_{n=0}^{\infty} (k*)^n g \end{aligned}$$

and thus,

$$\begin{aligned} K_{\varrho} g &= \chi_{\mathbb{R}_{\geq 0}}(\text{m}) (k' * + k(0)) \sum_{n=0}^{\infty} (k*)^n g \\ &= \chi_{\mathbb{R}_{\geq 0}}(\text{m}) (k' *) \sum_{n=0}^{\infty} (k*)^n g + k(0) \chi_{\mathbb{R}_{\geq 0}}(\text{m}) \sum_{n=1}^{\infty} (k*)^n g. \end{aligned}$$

By Lemma 1.3.5 we know that $k*, k' * : H_{\mu \vee \widetilde{\mu}}(\mathbb{R}; H) \rightarrow H_{\mu \vee \widetilde{\mu}}(\mathbb{R}; H)$ are bounded operators. Since by the choice of ϱ we have that $H_{\varrho}(\mathbb{R}_{\leq 0}; H) \hookrightarrow H_{\mu \vee \mu'}(\mathbb{R}; H)$, it follows that $K_{\varrho} g \in H_{\mu \vee \mu'}(\mathbb{R}_{\geq 0}; H)$ and that $K_{\varrho} : H_{\varrho}(\mathbb{R}_{\leq 0}; H) \rightarrow H_{\mu \vee \widetilde{\mu}}(\mathbb{R}_{\geq 0}; H)$ is bounded. Moreover, the computation above yields the independence of $\varrho \geq \mu \vee \widetilde{\mu}$. \square

In order to give a suitable subset of $\text{His}_{\varrho}(M, A)$, we also need to restrict the class of possible operators A .

Lemma 3.4.20. *Let $\varrho > \max\{s_0(M, A), 0, \mu, \widetilde{\mu}\}$ and assume that $(\partial_{0,\varrho} + A)^{-1} [H_{\varrho}(\mathbb{R}; H)] \subseteq H_{\varrho}^1(\mathbb{R}; H)$ and that k satisfies (3.16). Then, $S_{\varrho} : H_{\varrho}(\mathbb{R}; H) \rightarrow H_{\varrho}^1(\mathbb{R}; H)$ is well-defined and*

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bounded.

Proof. By the closed graph theorem, it suffices to check that $S_\varrho[H_\varrho(\mathbb{R}; H)] \subseteq H_\varrho^1(\mathbb{R}; H)$. So, let $f \in H_\varrho(\mathbb{R}; H)$ and set $u := S_\varrho f$. Then, using Lemma 3.4.18 we infer that

$$\begin{aligned} f &= \partial_{0,\varrho}(1 - k*)^{-1}u + Au \\ &= \partial_{0,\varrho}u + \partial_{0,\varrho} \sum_{n=1}^{\infty} (k*)^n u + Au \\ &= \partial_{0,\varrho}u + ((k'*) + k(0)) \sum_{n=0}^{\infty} (k*)^n u + Au \end{aligned}$$

and hence,

$$u = (\partial_{0,\varrho} + A)^{-1}(f - ((k'*) + k(0)) \sum_{n=0}^{\infty} (k*)^n u).$$

Since $u \in H_\varrho(\mathbb{R}; H)$, we derive that $f - ((k'*) + k(0)) \sum_{n=0}^{\infty} (k*)^n u \in H_\varrho(\mathbb{R}; H)$ and thus, $u \in H_\varrho^1(\mathbb{R}; H)$ by assumption. \square

Remark 3.4.21. The additional assumption imposed on A is called *maximal regularity* of the associated semigroup. It was proved in [deS64] that this is equivalent to the fact that $-A$ generates an analytic semigroup. Note that this characterization is false for Banach spaces [KL00]. For results in Banach spaces (more precisely UMD-spaces) we refer to [DV87, Wei01]. For an approach to maximal regularity for a class of evolutionary problems we refer to [PTW16b].

Corollary 3.4.22. *Let $\varrho > \max\{s_0(M, A), 0, \mu, \tilde{\mu}\}$ and assume that $(\partial_{0,\varrho} + A)^{-1}[H_\varrho(\mathbb{R}; H)] \subseteq H_\varrho^1(\mathbb{R}; H)$ and that k satisfies (3.16). Then each $g \in \chi_{\mathbb{R}_{\leq 0}}(\mathfrak{m})[H_\varrho^1(\mathbb{R}; H)]$ with $g(0-) \in D(A)$ belongs to $\text{His}_\varrho(M, A)$. Hence,*

$$X_\varrho^\nu = H \times H_\nu(\mathbb{R}_{\leq 0}; H)$$

for each $\nu \leq \varrho$.

Proof. Let $g \in \chi_{\mathbb{R}_{\leq 0}}(\mathfrak{m})[H_\varrho^1(\mathbb{R}; H)]$ with $g(0-) \in D(A)$. We compute

$$\begin{aligned} &S_\varrho(\delta_0 g(0-) - K_\varrho g) - \chi_{\mathbb{R}_{\geq 0}} g(0-) \\ &= S_\varrho(\delta_0 g(0-) - K_\varrho g - \partial_{0,\varrho}(1 - k*)^{-1} \chi_{\mathbb{R}_{\geq 0}} g(0-) - \chi_{\mathbb{R}_{\geq 0}} A g(0-)) \\ &= -S_\varrho \left((k' * + k(0)) \sum_{j=0}^{\infty} (k*)^j \chi_{\mathbb{R}_{\geq 0}} g(0-) + K_\varrho g + \chi_{\mathbb{R}_{\geq 0}} A g(0-) \right). \end{aligned}$$

The assertion now follows from Lemma 3.4.20. \square

Now we are in the position to prove, that we can associate a C_0 -semigroup on X_ϱ^ν for suitable ϱ and ν .

Proposition 3.4.23. *Let $\varrho > \max\{s_0(M, A), 0, \mu, \tilde{\mu}\}$. Assume that $(\partial_{0,\varrho} + A)^{-1}[H_\varrho(\mathbb{R}; H)] \subseteq H_\varrho^1(\mathbb{R}; H)$ and that k satisfies (3.16). Then $T_{(M,A)}$ extends to a C_0 -semigroup on $X_\varrho^\nu = H \times H_\nu(\mathbb{R}_{\leq 0}; H)$ for each $\max\{\mu, \tilde{\mu}\} \leq \nu \leq \varrho$.*

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Proof. Let $\max\{\mu, \tilde{\mu}\} \leq \nu \leq \varrho$. By Proposition 3.3.3 and Lemma 3.3.5 it suffices to prove that

$$T_{(M,A)}^{(1)} : D_\varrho \subseteq X_\varrho^\nu \rightarrow C_\varrho(\mathbb{R}_{\geq 0}; H)$$

is bounded. So let $g \in \text{His}_\varrho(M, A)$ and set $u := T_{(M,A)}^{(1)}(g(0-), g)$, i.e.

$$u = S_\varrho(\delta_0 g(0-) - K_\varrho g) = S_\varrho(\delta_0 g(0-) - K_\nu g).$$

We note that according to Proposition 3.4.19 $K_\nu : H_\nu(\mathbb{R}_{\leq 0}; H) \rightarrow H_{\mu \vee \tilde{\mu}}(\mathbb{R}_{\geq 0}; H) \hookrightarrow H_\varrho(\mathbb{R}; H)$ is bounded. By Lemma 3.4.20 we have that $S_\varrho : H_\varrho(\mathbb{R}; H) \rightarrow H_\varrho^1(\mathbb{R}; H)$ is bounded, which in turn gives that $S_\varrho : H_\varrho^{-1}(\mathbb{R}; H) \rightarrow H_\varrho(\mathbb{R}; H)$ is bounded. Thus, $u \in H_\varrho(\mathbb{R}; H)$ with

$$|u|_{H_\varrho(\mathbb{R}; H)} \leq C|(g(0-), g)|_{X_\varrho^\nu} \quad (3.17)$$

for some $C \geq 0$. Moreover, we have that

$$\begin{aligned} (\partial_{0,\varrho} + A)u + (k' * + k(0)) \sum_{n=0}^{\infty} (k*)^n u &= (\partial_{0,\varrho}(1 - k*)^{-1} + A)u \\ &= \delta_0 g(0-) - K_\nu g \end{aligned}$$

and hence

$$\begin{aligned} u &= (\partial_{0,\varrho} + A)^{-1} \left(\delta_0 g(0-) - K_\nu g - (k' * + k(0)) \sum_{n=0}^{\infty} (k*)^n u \right) \\ &= T_{-A} g(0-) - (\partial_{0,\varrho} + A)^{-1} \left(K_\nu g + (k' * + k(0)) \sum_{n=0}^{\infty} (k*)^n u \right), \end{aligned}$$

where T_{-A} denotes the contractive C_0 -semigroup generated by $-A$ (recall that A is m -accretive). Since $K_\nu g, u \in H_\varrho(\mathbb{R}; H)$ we have that

$$(\partial_{0,\varrho} + A)^{-1} \left(K_\nu g + (k' * + k(0)) \sum_{n=0}^{\infty} (k*)^n u \right) \in H_\varrho^1(\mathbb{R}; H) \hookrightarrow C_\varrho(\mathbb{R}; H),$$

according to Proposition 1.1.8 and thus, $u \in C_\varrho(\mathbb{R}_{\geq 0}; H)$ with

$$\begin{aligned} &|u|_{C_\varrho(\mathbb{R}_{\geq 0}; H)} \\ &\leq |g(0-)|_H + M \left| K_\nu g + (k' * + k(0)) \sum_{n=0}^{\infty} (k*)^n u \right|_{H_\varrho(\mathbb{R}; H)} \\ &\leq |g(0-)|_H + M \left(\|K_\nu\| \|g\|_{H_\nu(\mathbb{R}_{\leq 0}; H)} + (\|k'\|_{L_{1,\tilde{\mu}}} + \|k(0)\|) \frac{1}{1 - \|k\|_{L_{1,\mu}}} |u|_{H_\varrho(\mathbb{R}; H)} \right) \\ &\leq |g(0-)|_H + M \left(\|K_\nu\| \|g\|_{H_\nu(\mathbb{R}_{\leq 0}; H)} + (\|k'\|_{L_{1,\tilde{\mu}}} + \|k(0)\|) \frac{1}{1 - \|k\|_{L_{1,\mu}}} C|(g(0-), g)|_{X_\varrho^\nu} \right), \end{aligned}$$

where we have used (3.17) and where M denotes the operator norm of $(\partial_{0,\varrho} + A)^{-1} : H_\varrho(\mathbb{R}; H) \rightarrow H_\varrho^1(\mathbb{R}; H)$. \square

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We conclude this section by an estimate for the growth bound of $T_{(M,A)}$.

Proposition 3.4.24. *Let $\varrho > \max\{s_0(M, A), 0, \mu, \tilde{\mu}\}$. Assume that $(\partial_{0,\varrho} + A)^{-1} [H_\varrho(\mathbb{R}; H)] \subseteq H_\varrho^1(\mathbb{R}; H)$ and that k satisfies (3.16). Then $\omega(T_{(M,A)}) \leq \max\{s_0(M, A), \mu, \tilde{\mu}\}$.*

Proof. We need to show the assumptions of Theorem 3.3.7, that is we need to prove (GP-1) and (GP-2). We note that (GP-2) was already shown in Proposition 3.4.19 and thus, it suffices to check (GP-1). For doing so, let $\lambda > 0$, $\varrho > \max\{s_0(M, A), \mu, \tilde{\mu}\}$ and consider the mapping

$$\Phi : \mathbb{C}_{\operatorname{Re} > \varrho} \ni z \mapsto zM(z) - (z + \lambda)M(z + \lambda) \in L(H).$$

We compute

$$\begin{aligned} \Phi(z) &= z(1 - \sqrt{2\pi}\widehat{k}(z))^{-1} - (z + \lambda)(1 - \sqrt{2\pi}\widehat{k}(z + \lambda))^{-1} \\ &= \sqrt{2\pi}z(1 - \sqrt{2\pi}\widehat{k}(z))^{-1}(\widehat{k}(z + \lambda) - \widehat{k}(z)) \left(1 - \sqrt{2\pi}\widehat{k}(z + \lambda)\right)^{-1} - \lambda(1 - \sqrt{2\pi}\widehat{k}(z + \lambda))^{-1}. \end{aligned}$$

Since

$$\|(1 - \sqrt{2\pi}\widehat{k}(z))^{-1}\| \leq \frac{1}{1 - |k|_{L_{1,\mu}}} \quad (z \in \mathbb{C}_{\operatorname{Re} > \varrho}),$$

it suffices to prove that $z \mapsto z(\widehat{k}(z + \lambda) - \widehat{k}(z))$ is bounded on $\mathbb{C}_{\operatorname{Re} > \varrho}$. Using Lemma 3.4.18 we have that

$$\begin{aligned} z(\widehat{k}(z + \lambda) - \widehat{k}(z)) &= (z + \lambda)\widehat{k}(z + \lambda) - z\widehat{k}(z) - \lambda\widehat{k}(z + \lambda) \\ &= \widehat{k}'(z + \lambda) - \widehat{k}'(z) - \lambda\widehat{k}(z + \lambda), \end{aligned}$$

which yields that

$$\|z(\widehat{k}(z + \lambda) - \widehat{k}(z))\| \leq 2|k'|_{L_{1,\tilde{\mu}}} + \lambda|k|_{L_{1,\mu}} \quad (z \in \mathbb{C}_{\operatorname{Re} > \varrho}).$$

This completes the proof. \square

Remark 3.4.25. The latter proposition yields the exponential stability of $T_{(M,A)}$, provided that $s_0(M, A), \mu, \tilde{\mu} < 0$. For sufficient assumptions on the kernel k and the operator A , which yield $s_0(M, A) < 0$ we refer to 2.3.1.

3.5. Notes

We have provided a way to incorporate initial values and histories in the framework of evolutionary problems. The key observation was that suitable right-hand sides belonging to the extrapolation space H_ϱ^{-1} together with the causality of the solution operator, yield that the solution indeed satisfies the desired initial conditions. This idea was already used in [PM11], however just for the case of pure initial values. A similar way to incorporate histories in the case of delay equations was suggested in [KPS⁺14]. The main improvement of the strategy presented here, is that we are able to define a space of “admissible” initial values and histories for a given evolutionary problem. Here, admissible means that these spaces of initial values

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and histories allow for the definition of a C_0 -semigroup. To the best of the author's knowledge, so far there was no intrinsic way to define such spaces in that generality. Of course, for each of the classes of differential equations discussed in Section 3.4, suitable choices for those initial values were known and the theory of C_0 -semigroups was already established for those.

Indeed, for the case of differential-algebraic equations, the notion of *consistent* initial values is well-established (see e.g. [KM06] in finite dimensions or [Rei07] for a class of DAEs in infinite dimensions). As we have shown in Example 3.4.9, these consistent initial values coincide with our space $\text{IV}_\varrho(M, A)$. It should be noted that in the theory of differential-algebraic equations it is common to assume that the corresponding evolutionary problem is *not* well-posed in the sense that the solution operator S_ϱ is bounded from H_ϱ to H_ϱ , but just bounded from H_ϱ to H_ϱ^{-k} for some $k \in \mathbb{N}$. Thus, those problems are not covered by our abstract results so far and it is postponed to future studies to investigate, how our results could be generalized to evolutionary problems whose solution operators are just bounded from H_ϱ to H_ϱ^{-k} . Moreover, we have restricted ourselves to homogeneous problems, i.e. evolutionary problems of the form

$$(\partial_{0,\varrho} M(\partial_{0,\varrho}) + A) u = 0 \text{ on } \mathbb{R}_{>0}.$$

It is well-known that in general a non-vanishing source term on the right-hand side has an influence on the possible choices of initial values. Again, the study of those problems is postponed to future work.

In the theory of delay equations there are several choices for “admissible” histories. We just mention [Web76, BP05], where the histories belong to some L_p -space, or [Hal71, HL93] for histories in the space of continuous functions. Especially in the theory of nonlinear delay differential equations and equations with state dependent delay, a restriction of admissible histories is needed to obtain a solution. For this topic we refer to [Rue09] for nonlinear problems and to [Wal03, Wal09] for state dependent delay equations.

Finally, there exists a large amount of articles considering C_0 -semigroups associated with integro-differential equations. We just mention [GLS90] for finite dimensions and [BP05, KS83, CG82] for infinite dimensions. Another approach to integro-differential equations is provided in [Prü93], where not a C_0 -semigroup is associated to the problem but a so-called resolvent family, which allows for a slightly weaker notion of solutions. This approach was successfully applied to several problems.

A further class of differential equations, which was not addressed in Section 3.4 is the class of fractional differential equations. Although, such equations are covered by the framework of evolutionary problems (see [PTW15a]) they do not seem to fit in the framework developed in this chapter. The main reason is that the space $\text{His}_\varrho(M, A)$ does not seem to be the right choice for those problems. Indeed, since the Sobolev embedding theorem also holds for H_ϱ^α with $\alpha > \frac{1}{2}$ one could weaken the definition of $\text{His}_\varrho(M, A)$ in the sense that the solution should belong to some H_ϱ^α for $\alpha > \frac{1}{2}$. So it would be a valuable project to inspect, how the framework could be generalized in order to cover fractional differential equations.

We conclude this section by an open problem. In Theorem 3.3.7 we proved that $\omega(T_{(M,A)}) \leq s_0(M, A)$ for suitable material laws M . As it was already pointed out, this result reduces to the famous Gearhart-Prüß Theorem if $M = 1$. However, in this case it is obvious that even equality holds, since $s_0(1, A) \leq \omega(T_{(1,A)})$ holds trivially. So the question arises, whether there are examples of material laws M , where the estimate $\omega(T_{(M,A)}) < s_0(M, A)$ holds. There are good reasons to believe that there is such an example, even in the simple case $M(z) = M_0 + z^{-1}M_1$ with a non-invertible M_0 , since in this case $\text{IV}_\varrho(M, A)$ is not dense in

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H in general. Thus, the corresponding semigroup acts on a proper subspace of H , while the abscissa of boundedness $s_0(M, A)$ is defined using the whole space H , and hence, the growth bound could be strictly less than the abscissa of boundedness. However, so far the author was not able to construct an example.

A. The Fourier-Laplace transform

In this section we briefly recall some well-known facts about the Fourier- and the Fourier-Laplace transform. We present the statements just for scalar-valued functions, although we will use them in the Hilbert space-valued case. However, employing the tensor product structure of $L_2(\mathbb{R}; H, \mu) = L_2(\mathbb{R}, \mu) \otimes H$ for a Hilbert space H and a Borel-measure μ , we immediately get that the results carry over to the Hilbert space case. For the theory of tensor products of Hilbert spaces and operators we refer to [Wei80, Ber86], where the case of selfadjoint operators is considered, and to [PM11, Tro11] for the general case.

We start our considerations by studying the Fourier transform.

Definition. Let $f \in L_1(\mathbb{R})$. Then we define the *Fourier transform* $\mathcal{F}f$ of f by

$$(\mathcal{F}f)(x) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ixt} f(t) dt \quad (x \in \mathbb{R}).$$

Remark A.1. Obviously, $\mathcal{F}f$ is continuous (by dominated convergence) and bounded by $\frac{1}{\sqrt{2\pi}}\|f\|_{L_1}$, and hence, the Fourier transform \mathcal{F} is a bounded linear operator from $L_1(\mathbb{R})$ to $C_b(\mathbb{R})$. Choosing $f \in C_c^\infty(\mathbb{R})$ one obtains $\mathcal{F}f \in C_0(\mathbb{R})$. Indeed, one has

$$\mathcal{F}f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ixt} f(t) dt = \frac{1}{ix} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ixt} f'(t) dt \quad (x \neq 0)$$

and hence,

$$\limsup_{|x| \rightarrow \infty} |\mathcal{F}f(x)| \leq \limsup_{|x| \rightarrow \infty} \frac{1}{|x|} \frac{1}{\sqrt{2\pi}} \|f'\|_{L_1} = 0.$$

By continuous extension, we thus get $\mathcal{F}f \in C_0(\mathbb{R})$ for each $f \in L_1(\mathbb{R})$. This statement is known as the Lemma of Riemann-Lebesgue.

Next, we will consider the Fourier transform on a particular subspace of $L_1(\mathbb{R})$, namely the Schwartz space $\mathcal{S}(\mathbb{R})$ of rapidly decreasing functions.

Proposition A.2. *The Fourier transform is a bijection on $\mathcal{S}(\mathbb{R})$, i.e. $\mathcal{F}|_{\mathcal{S}(\mathbb{R})} : \mathcal{S}(\mathbb{R}) \rightarrow \mathcal{S}(\mathbb{R})$ is bijective. Moreover, for $f \in L_1(\mathbb{R})$ with $\mathcal{F}f \in L_1(\mathbb{R})$ we have*

$$f(x) = (\mathcal{F}^* \mathcal{F}f)(x) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ixt} \mathcal{F}f(t) dt = \mathcal{F}f(-x) \quad (x \in \mathbb{R} \text{ a.e.}), \quad (\text{A.1})$$

and we have for all $f, g \in L_1(\mathbb{R})$:

$$\int_{\mathbb{R}} (\mathcal{F}f)(x)^* g(x) dx = \int_{\mathbb{R}} f(x)^* (\mathcal{F}^* g)(x) dx. \quad (\text{A.2})$$

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Proof. Let $\phi \in \mathcal{S}(\mathbb{R})$. We first show that $\mathcal{F}\phi \in \mathcal{S}(\mathbb{R})$. For doing so, let $k, n \in \mathbb{N}$ and compute

$$\begin{aligned} |x|^n \left| \left(\partial^k \mathcal{F}\phi \right) (x) \right| &= \frac{1}{\sqrt{2\pi}} \left| \int_{\mathbb{R}} (-ix)^n (-it)^k e^{-ixt} \phi(t) dt \right| \\ &= \frac{1}{\sqrt{2\pi}} \left| \int_{\mathbb{R}} (-it)^k e^{-ixt} \phi^{(n)}(t) dt \right| \\ &\leq \frac{1}{\sqrt{2\pi}} \left\| (t \mapsto t^k \phi^{(n)}(t)) \right\|_{L_1} \end{aligned}$$

for each $x \in \mathbb{R}$, where we have used differentiation under the integral and integration by parts. The latter estimate shows $\mathcal{F}\phi \in \mathcal{S}(\mathbb{R})$. In order to show (A.1), we first prove (A.2). Let $f, g \in L_1(\mathbb{R})$. Then we have by Fubini's theorem

$$\begin{aligned} \int_{\mathbb{R}} (\mathcal{F}f)(x)^* g(x) dx &= \int_{\mathbb{R}} \left(\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ixt} f(t) dt \right)^* g(x) dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{ixt} f(t)^* g(x) dt dx \\ &= \int_{\mathbb{R}} f(t)^* (\mathcal{F}^*g)(t) dt. \end{aligned}$$

Consider now $\gamma \in \mathcal{S}(\mathbb{R})$ given by

$$\gamma(x) := e^{-\frac{x^2}{2}} \quad (x \in \mathbb{R}).$$

Then, $\gamma(0) = 1$ and $\gamma'(x) + x\gamma(x) = 0$ for each $x \in \mathbb{R}$. Moreover, $\mathcal{F}\gamma(0) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{x^2}{2}} dx = 1$ and

$$\begin{aligned} (\mathcal{F}\gamma)'(x) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} (-it) e^{-ixt} \gamma(t) dt \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ixt} i \gamma'(t) dt \\ &= -\frac{1}{\sqrt{2\pi}} x \int_{\mathbb{R}} e^{-ixt} \gamma(t) dt = -x \mathcal{F}\gamma(x) \end{aligned}$$

for each $x \in \mathbb{R}$. Thus, $\mathcal{F}\gamma$ and γ are both solutions of the initial value problem $y'(x) + xy(x) = 0$ and $y(0) = 1$, and so, they coincide. Let now $x \in \mathbb{R}, a > 0$ and $f \in L_1(\mathbb{R})$ with $\mathcal{F}f \in L_1(\mathbb{R})$. Since $\mathcal{F}^*f = \mathcal{F}\sigma_{-1}f$, where $(\sigma_{-1}f)(x) := f(-x)$, we compute, using (A.2),

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ixt} \gamma(at) (\mathcal{F}f)(t) dt = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} (e^{-ixt} \gamma(at))^* (\mathcal{F}^*\sigma_{-1}f)(t) dt$$

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$$\begin{aligned}
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} (\mathcal{F} e^{-ix \cdot} \gamma(a \cdot)) (t)^* f(-t) dt \\
&= \frac{1}{2\pi} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} e^{-ity} e^{-ixy} \gamma(ay) dy \right)^* f(-t) dt \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i(t-x)y} \gamma(ay) dy f(t) dt \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(t) \frac{1}{a} \mathcal{F} \gamma \left(\frac{t-x}{a} \right) dt \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(az+x) \gamma(z) dz. \tag{A.3}
\end{aligned}$$

By dominated convergence we obtain

$$\lim_{a \rightarrow 0} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ixt} \gamma(at) (\mathcal{F} f)(t) dt = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} (\mathcal{F} f)(t) e^{ixt} dt. \tag{A.4}$$

For the term on the right hand side of (A.3), we consider the linear operators

$$\begin{aligned}
S_a : L_1(\mathbb{R}) &\rightarrow L_1(\mathbb{R}) \\
f &\mapsto \left(x \mapsto \int_{\mathbb{R}} f(az+x) \gamma(z) dz \right).
\end{aligned}$$

Then

$$|S_a f|_{L_1} = \int_{\mathbb{R}} \left| \int_{\mathbb{R}} f(az+x) \gamma(z) dz \right| dx \leq \int_{\mathbb{R}} \int_{\mathbb{R}} |f(az+x)| dx \gamma(z) dz \leq |f|_{L_1} |\gamma|_{L_1},$$

and so $\|S_a\| \leq |\gamma|_{L_1}$ for each $a > 0$. Moreover, since $S_a \psi \rightarrow \psi |\gamma|_{L_1}$ in $L_1(\mathbb{R})$ as $a \rightarrow 0$ for each $\psi \in C_c^\infty(\mathbb{R})$, we deduce $S_a f \rightarrow f |\gamma|_{L_1}$ in $L_1(\mathbb{R})$ as $a \rightarrow 0$ for each $f \in L_1(\mathbb{R})$. Thus, choosing a suitable sequence $(a_n)_{n \in \mathbb{N}}$ of positive reals with $a_n \rightarrow 0$, we get that

$$\int_{\mathbb{R}} f(a_n z + x) \gamma(z) dz \rightarrow f(x) |\gamma|_{L_1} \quad (n \rightarrow \infty)$$

for almost every $x \in \mathbb{R}$. Thus, (A.3), (A.4) and $|\gamma|_{L_1} = \sqrt{2\pi}$ imply

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ixt} (\mathcal{F} f)(t) dt = f(x)$$

for almost every $x \in \mathbb{R}$, which is (A.1). In particular, we obtain for $\phi \in \mathcal{S}(\mathbb{R})$

$$\phi(x) = (\mathcal{F}^* \mathcal{F} \phi)(x) \quad (x \in \mathbb{R}),$$

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which gives that $\mathcal{F}|_{\mathcal{S}(\mathbb{R})}$ is one-to-one. Moreover,

$$(\mathcal{F}\mathcal{F}^*\phi)(x) = (\mathcal{F}\mathcal{F}\sigma_{-1}\phi)(x) = (\mathcal{F}^*\mathcal{F}\sigma_{-1}\phi)(-x) = \phi(x) \quad (x \in \mathbb{R}),$$

which shows that $\mathcal{F}|_{\mathcal{S}(\mathbb{R})}$ is onto. \square

The latter proposition yields that \mathcal{F} can be established as a unitary operator on $L_2(\mathbb{R})$.

Theorem A.3 (Plancherel). *The operator $\mathcal{F}|_{\mathcal{S}(\mathbb{R})} : \mathcal{S}(\mathbb{R}) \subseteq L_2(\mathbb{R}) \rightarrow L_2(\mathbb{R})$ has a unique unitary extension, again denoted by $\mathcal{F} : L_2(\mathbb{R}) \rightarrow L_2(\mathbb{R})$.*

Proof. By Proposition A.2 we have $\mathcal{F}[\mathcal{S}(\mathbb{R})] = \mathcal{S}(\mathbb{R})$, which shows, that $\mathcal{F}|_{\mathcal{S}(\mathbb{R})}$ has dense range. Moreover, by (A.2) we have for $\phi \in \mathcal{S}(\mathbb{R})$

$$|\mathcal{F}\phi|_{L_2(\mathbb{R})}^2 = \langle \mathcal{F}\phi | \mathcal{F}\phi \rangle_{L_2(\mathbb{R})} = \langle \phi | \mathcal{F}^* \mathcal{F} \phi \rangle_{L_2(\mathbb{R})} = |\phi|_{L_2(\mathbb{R})}^2,$$

which shows, that $\mathcal{F}|_{\mathcal{S}(\mathbb{R})}$ is an isometry on $L_2(\mathbb{R})$. Both things imply that $\mathcal{F}|_{\mathcal{S}(\mathbb{R})}$ has a unique unitary extension. \square

Definition. Let $f \in L_{1,\varrho}(\mathbb{R}) := \{g : \mathbb{R} \rightarrow \mathbb{C}; g \text{ measurable, } \int_{\mathbb{R}} |g| e^{-\varrho t} dt < \infty\}$ for some $\varrho \in \mathbb{R}$. Then we define

$$\mathcal{L}_{\varrho} f(t) := \mathcal{F}(e^{-\varrho m} f)(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-(it+\varrho)x} f(x) dx \quad (t \in \mathbb{R}),$$

the *Fourier-Laplace transform* of f . Moreover, we introduce the notation

$$\widehat{f}(it + \varrho) := \mathcal{L}_{\varrho} f(t) \quad (t \in \mathbb{R}).$$

Corollary A.4. *Let $\varrho \in \mathbb{R}$. Then the Fourier-Laplace transform can be established as a unitary operator $\mathcal{L}_{\varrho} : H_{\varrho}(\mathbb{R}) \rightarrow L_2(\mathbb{R})$.*

Proof. We note that $\mathcal{L}_{\varrho} = \mathcal{F} e^{-\varrho m}$. Since $e^{-\varrho m} : H_{\varrho}(\mathbb{R}) \rightarrow L_2(\mathbb{R})$ is obviously unitary, we derive that \mathcal{L}_{ϱ} is unitary by Theorem A.3. \square

We also provide an adapted version of the Riemann-Lebesgue Lemma.

Proposition A.5. *Let $f \in L_1(\mathbb{R}_{\geq 0})$, such that $(t \mapsto tf(t)) \in L_1(\mathbb{R}_{\geq 0})$. Moreover, let $K \subseteq \mathbb{R}_{\geq 0}$ compact. Then*

$$\forall \varepsilon > 0 \exists M > 0 \forall |t| \geq M, \varrho \in K : |\mathcal{L}_{\varrho} f(t)| < \varepsilon.$$

Proof. We first show, that

$$\mathbb{R}_{\geq 0} \ni \varrho \mapsto \mathcal{L}_{\varrho} f \in C_b(\mathbb{R})$$

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is Lipschitz-continuous. Indeed, for $\varrho, \varrho' \in \mathbb{R}_{\geq 0}$ we have that

$$\begin{aligned} |\mathcal{L}_\varrho f(t) - \mathcal{L}_{\varrho'} f(t)| &\leq \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}_{\geq 0}} |f(s)| \left| e^{-\varrho s} - e^{-\varrho' s} \right| ds \\ &\leq \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}_{\geq 0}} |s f(s)| ds |\varrho - \varrho'| \end{aligned}$$

for each $t \in \mathbb{R}$, which shows the Lipschitz-continuity. Consequently,

$$\{\mathcal{L}_\varrho f; \varrho \in K\} \subseteq C_b(\mathbb{R})$$

is compact. Let now $\varepsilon > 0$ and choose $K' \subseteq K$ finite such that

$$\{\mathcal{L}_\varrho f; \varrho \in K\} \subseteq \bigcup_{\varrho' \in K'} B(\mathcal{L}_{\varrho'} f, \varepsilon).$$

According to the Lemma of Riemann-Lebesgue (Remark A.1), there is some $M > 0$ such that

$$\forall |t| \geq M, \varrho' \in K' : |\mathcal{L}_{\varrho'} f(t)| < \varepsilon.$$

Let now $\varrho \in K, |t| \geq M$. Then there is $\varrho' \in K'$ such that $\|\mathcal{L}_\varrho f - \mathcal{L}_{\varrho'} f\|_\infty < \varepsilon$ and we conclude

$$|\mathcal{L}_\varrho f(t)| \leq \|\mathcal{L}_\varrho f - \mathcal{L}_{\varrho'} f\| + |\mathcal{L}_{\varrho'} f(t)| < 2\varepsilon. \quad \square$$

As a further property of the Fourier-Laplace transform we want to state a theorem of Paley and Wiener, characterizing the L_2 -functions supported on the positive real axis by their Fourier-Laplace transform. We begin with the following lemma.

Lemma A.6. *Let $f \in \bigcap_{\varrho > 0} H_\varrho(\mathbb{R})$ with $\sup_{\varrho > 0} |f|_{H_\varrho} < \infty$. Then $f \in L_2(\mathbb{R}_{\geq 0})$ and $|f|_{L_2} = \sup_{\varrho > 0} |f|_{H_\varrho} = \lim_{\varrho \rightarrow 0} |f|_{H_\varrho}$.*

Proof. We first show that f is supported on $\mathbb{R}_{\geq 0}$. For doing so, let $a < b < 0$. Then for each $\varrho > 0$ we estimate

$$\begin{aligned} \int_a^b |f(t)| dt &= \int_a^b e^{\varrho t} e^{-\varrho t} |f(t)| dt \\ &\leq \left(\int_a^b e^{2\varrho t} dt \right)^{\frac{1}{2}} |f|_{H_\varrho} \\ &= \left(\frac{1}{2\varrho} (e^{2\varrho b} - e^{2\varrho a}) \right)^{\frac{1}{2}} |f|_{H_\varrho} \\ &\leq e^{\varrho b} \sqrt{(b-a)} |f|_{H_\varrho}, \end{aligned}$$

by the mean value theorem. Letting $\varrho \rightarrow \infty$ and using that $b < 0$ as well as $\sup_{\varrho > 0} |f|_{H_\varrho} < \infty$

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we derive

$$\int_a^b |f(t)| \, dt = 0.$$

As $a < b < 0$ were chosen arbitrarily, we get $\text{spt } f \subseteq \mathbb{R}_{\geq 0}$. Moreover, we estimate for each $n \in \mathbb{N}$ and $\varrho > 0$

$$\int_0^n |f(t)|^2 \, dt = \int_0^n e^{2\varrho t} e^{-2\varrho t} |f(t)|^2 \, dt \leq e^{2\varrho n} |f|_{H_\varrho}^2 \leq e^{2\varrho n} \sup_{\mu > 0} |f|_{H_\mu}^2.$$

Letting $\varrho \rightarrow 0$ we derive $\int_0^n |f(t)|^2 \, dt \leq \sup_{\mu > 0} |f|_{H_\mu}^2$ for each $n \in \mathbb{N}$ and so, $f \in L_2(\mathbb{R}_{\geq 0})$ with $|f|_{L_2} \leq \sup_{\varrho > 0} |f|_{H_\varrho}$. Moreover, since $\text{spt } f \subseteq \mathbb{R}_{\geq 0}$ we get $|f|_{H_\varrho} \leq |f|_{H_\mu}$ for $\varrho > \mu \geq 0$, and hence, $|f|_{L_2} \geq \sup_{\varrho > 0} |f|_{H_\varrho} = \lim_{\varrho \rightarrow 0} |f|_{H_\varrho}$, which completes the proof. \square

We are now able to state the Paley-Wiener Theorem. We mainly follow the proof presented in [Rud87, 19.2 Theorem].

Theorem A.7 (Paley-Wiener, [PW34]). *Let $g \in \mathcal{H}^2(\mathbb{C}_{>0})$. Then there exists $f \in L_2(\mathbb{R}_{\geq 0})$ such that*

$$g(it + \varrho) = \widehat{f}(it + \varrho) = (\mathcal{L}_\varrho f)(t) \quad (t \in \mathbb{R}, \varrho > 0).$$

Moreover, $|f|_{L_2} = |g|_{\mathcal{H}^2}$.

Proof. For $\varrho > 0$ we define $g_\varrho = (t \mapsto g(it + \varrho))$. By assumption, $g_\varrho \in L_2(\mathbb{R})$ for each $\varrho > 0$ and $\sup_{\varrho > 0} |g_\varrho|_{L_2} < \infty$. For $\varrho > 0$ we define $f_\varrho := \mathcal{F}^* g_\varrho \in L_2(\mathbb{R})$. Moreover, we set $f(x) := e^x f_1(x)$ for $x \in \mathbb{R}$. We prove that $f \in \bigcap_{\varrho > 0} H_\varrho(\mathbb{R})$. For doing so, let $\varrho > 0$. Moreover we fix $a > 0$ and $x \in \mathbb{R}$. By Cauchy's integral theorem we have

$$\begin{aligned} i \int_{-a}^a e^{(it+1)x} g(it+1) \, dt - \int_{\varrho}^1 e^{(ia+\kappa)x} g(ia+\kappa) \, d\kappa - i \int_{-a}^a e^{(it+\varrho)x} g(it+\varrho) \, dt + \\ + \int_{\varrho}^1 e^{(-ia+\kappa)x} g(-ia+\kappa) \, d\kappa = 0. \end{aligned} \quad (\text{A.5})$$

Moreover, since

$$\int_{\mathbb{R}} \left| \int_{\varrho}^1 e^{(\pm ia+\kappa)x} g(\pm ia+\kappa) \, d\kappa \right|^2 \, da \leq |1 - \varrho| |g|_{\mathcal{H}^2}^2 \int_{\varrho}^1 e^{2\kappa x} \, d\kappa < \infty$$

we find a sequence $(a_n)_{n \in \mathbb{N}}$ such that $a_n \rightarrow \infty$ and

$$\int_{\varrho}^1 e^{(\pm ia_n+\kappa)x} g(\pm ia_n+\kappa) \, d\kappa \rightarrow 0 \quad (n \rightarrow \infty).$$

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Hence, (A.5) implies

$$\int_{-a_n}^{a_n} e^{(it+1)x} g(it+1) dt - \int_{-a_n}^{a_n} e^{(it+\varrho)x} g(it+\varrho) dt \rightarrow 0 \quad (n \rightarrow \infty).$$

Since on the other hand we have

$$\lim_{n \rightarrow \infty} \int_{-a_n}^{a_n} e^{itx} g(it+\mu) dt = \lim_{n \rightarrow \infty} \mathcal{F}^* (\chi_{[-a_n, a_n]}(m) g_\mu) \rightarrow f_\mu \quad \text{in } L_2(\mathbb{R})$$

for each $\mu > 0$, we may choose a subsequence (again labeled by n), such that

$$\int_{-a_n}^{a_n} e^{(it+1)x} g(it+1) dt - \int_{-a_n}^{a_n} e^{(it+\varrho)x} g(it+\varrho) dt \rightarrow e^x f_1(x) - e^{\varrho x} f_\varrho(x) \quad (n \rightarrow \infty)$$

for almost every $x \in \mathbb{R}$. Thus, we have

$$f(x) = e^x f_1(x) = e^{\varrho x} f_\varrho(x) \quad (x \in \mathbb{R} \text{ a.e.})$$

and since $f_\varrho \in L_2(\mathbb{R})$ we obtain $f \in H_\varrho(\mathbb{R})$. Moreover, the latter equality together with Theorem A.3 gives

$$\sup_{\varrho > 0} |f|_{H_\varrho} = \sup_{\varrho > 0} |f_\varrho|_{L_2} = \sup_{\varrho > 0} |g_\varrho|_{L_2} = |g|_{\mathcal{H}^2}.$$

The assertion now follows from Lemma A.6. □

As a consequence of the latter theorem we obtain the following corollary.

Corollary A.8. *Let $\varrho_0 \in \mathbb{R}$. Then the mapping*

$$\begin{aligned} \mathcal{L} : H_{\varrho_0}(\mathbb{R}_{\geq 0}) &\rightarrow \mathcal{H}^2(\mathbb{C}_{\operatorname{Re} > \varrho_0}) \\ f &\mapsto \widehat{f} \end{aligned}$$

is unitary.

Proof. Let $f \in H_{\varrho_0}(\mathbb{R}_{\geq 0})$. Then $(t \mapsto e^{-\varrho t} f(t)) \in L_1(\mathbb{R}_{\geq 0}) \cap L_2(\mathbb{R}_{\geq 0})$ for each $\varrho > \varrho_0$ and hence

$$\widehat{f}(z) = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-zs} f(s) ds \quad (z \in \mathbb{C}_{\operatorname{Re} > \varrho_0}).$$

From this representation, we see that \widehat{f} is analytic. Moreover, due to Theorem A.3 we have

$$|\widehat{f}(i \cdot + \varrho)|_{L_2} = |e^{-\varrho \cdot} f|_{L_2} = |f|_{H_\varrho} \quad (\varrho > \varrho_0).$$

Moreover, since $\operatorname{spt} f \subseteq \mathbb{R}_{\geq 0}$ we have that $\varrho \mapsto |f|_{H_\varrho}$ is monotone decreasing and by monotone

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convergence we obtain $|f|_{H_{\varrho_0}} = \lim_{\varrho \rightarrow \varrho_0} |f|_{H_\varrho} = \sup_{\varrho > \varrho_0} |f|_{H_\varrho}$. Thus, we have

$$|\widehat{f}|_{\mathcal{H}^2} = \sup_{\varrho > \varrho_0} |\widehat{f}(\mathbf{i} \cdot + \varrho)|_{L_2} = \sup_{\varrho > \varrho_0} |f|_{H_\varrho} = |f|_{H_{\varrho_0}},$$

which shows that \mathcal{L} is isometric. To show that it is onto, let $g \in \mathcal{H}^2(\mathbb{C}_{\operatorname{Re} > \varrho_0})$. Then we define $\widetilde{g} := g(\cdot + \varrho_0) \in \mathcal{H}^2(\mathbb{C}_{\operatorname{Re} > 0})$ and by Theorem A.7 there is $h \in L_2(\mathbb{R}_{\geq 0})$ such that $\widehat{h} = \widetilde{g}$ on $\mathbb{C}_{\operatorname{Re} > 0}$. Setting $f := e^{\varrho_0 \cdot} h \in H_{\varrho_0}(\mathbb{R}_{\geq 0})$ we obtain

$$\begin{aligned} \widehat{f}(\mathbf{i}t + \varrho) &= \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-(\mathbf{i}t + \varrho)s} e^{\varrho_0 s} h(s) \, ds \\ &= \widehat{h}(\mathbf{i}t + \varrho - \varrho_0) = \widetilde{g}(\mathbf{i}t + \varrho - \varrho_0) = g(\mathbf{i}t + \varrho) \quad (t \in \mathbb{R}, \varrho > \varrho_0), \end{aligned}$$

which finishes the proof. \square

Using this result, we are able to prove the following statement, which will be used later in the appendix.

Corollary A.9. *Let $\Lambda \subseteq \mathbb{R}_{>0}$ be a set with an accumulation point. Then the set $\{e^{-\lambda \cdot}; \lambda \in \Lambda\}$ is total in $L_1(\mathbb{R}_{\geq 0})$.*

Proof. Since $L_1(\mathbb{R}_{\geq 0})' = L_\infty(\mathbb{R}_{\geq 0})$, it suffices to show that if $f \in L_\infty(\mathbb{R}_{\geq 0})$ with

$$\int_0^\infty e^{-\lambda t} f(t) \, dt = 0 \quad (\lambda \in \Lambda),$$

then $f = 0$. Since $L_\infty(\mathbb{R}_{\geq 0}) \subseteq \bigcap_{\varrho > 0} H_\varrho(\mathbb{R}_{\geq 0})$, we get by the latter equality $\widehat{f}(\lambda) = 0$ for each $\lambda \in \Lambda$. Since \widehat{f} is holomorphic on $\mathbb{C}_{\operatorname{Re} > 0}$ and zero on a set with accumulation point, it follows that $\widehat{f} = 0$ and hence, by Corollary A.8, we obtain $f = 0$. \square

B. The Theorem of Rademacher

In this section we prove Rademacher's Theorem for Lipschitz-continuous functions on \mathbb{R} with values in a reflexive Banach space. The theorem, proved by Rademacher in 1919 ([Rad19]) for functions $F : U \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m$, states that a Lipschitz-continuous function is differentiable almost everywhere and its derivative is bounded.

First, we begin to prove the scalar-valued case. For doing so, we need Lebesgue's famous differentiation theorem, which will be shown in the first part of this appendix. We follow the argumentation presented in [Rud87, Chapter 7] and we start by stating Vitalis Covering Lemma. For doing so, we need the following auxiliary result.

Lemma B.1. *Let X be a metric space and $\mathcal{U} \subseteq \mathcal{P}(X)$ a collection of subsets with non-empty interior. Then there exists a maximal disjoint subcollection $\mathcal{S} \subseteq \mathcal{U}$ (i.e. all elements in \mathcal{S} are pairwise disjoint). Moreover, if X is separable, then \mathcal{S} is at most countable.*

Proof. We apply Zorn's Lemma to $\mathcal{M} := \{\mathcal{S} \subseteq \mathcal{U} ; \forall U, V \in \mathcal{S} : U \cap V \neq \emptyset \Rightarrow U = V\}$ ordered by inclusion. Clearly, every totally ordered subset of \mathcal{M} has an upper bound given by its union and hence, the first assertion follows. Assume now that X is separable and let $(x_n)_{n \in \mathbb{N}}$ be a dense sequence in X . Then the mapping

$$F : \{x_n ; n \in \mathbb{N}\} \cap \bigcup \mathcal{S} \rightarrow \mathcal{S} \\ x \mapsto U(x),$$

where $U(x) \in \mathcal{S}$ is the unique set with $x \in U(x)$ (recall that \mathcal{S} consists of pairwise disjoint sets), is onto. Indeed, for each $U \in \mathcal{S}$ there exists $n \in \mathbb{N}$ with $x_n \in \overset{\circ}{U}$, since $\overset{\circ}{U}$ is non-empty and $\{x_n ; n \in \mathbb{N}\}$ is dense. Hence, \mathcal{S} is at most countable. \square

Proposition B.2 (Vitali Covering Lemma). *Let $n \in \mathbb{N}_{>0}$, $\mathcal{B} \subseteq \mathcal{P}(\mathbb{R}^n)$ a collection of non-empty balls such that*

$$R := \sup_{U \in \mathcal{B}} \text{diam } U < \infty.$$

Then there exists an at most countable disjoint subcollection $\tilde{\mathcal{B}} \subseteq \mathcal{B}$ such that

$$\lambda\left(\bigcup \mathcal{B}\right) \leq 5^n \lambda\left(\bigcup \tilde{\mathcal{B}}\right).$$

Proof. For $k \in \mathbb{N}$ we define

$$\mathcal{B}_k := \left\{ U \in \mathcal{B} ; 2^{-(k+1)}R < \text{diam } U \leq 2^{-k}R \right\}.$$

We set $\tilde{\mathcal{B}}_0 \subseteq \mathcal{B}_0$ as a maximal disjoint subcollection and define recursively $\tilde{\mathcal{B}}_{k+1}$ to be a

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maximal disjoint subcollection of

$$\mathcal{C}_{k+1} := \left\{ U \in \mathcal{B}_{k+1}; \forall V \in \bigcup_{j=0}^k \tilde{\mathcal{B}}_j : U \cap V = \emptyset \right\}.$$

Then $\tilde{\mathcal{B}} := \bigcup_{k \in \mathbb{N}} \tilde{\mathcal{B}}_k$ is a disjoint subcollection of \mathcal{B} , which is at most countable by Lemma B.1. Moreover, for each $U \in \mathcal{B}_k$ for some $k \in \mathbb{N}$, there exists $V \in \bigcup_{j=0}^k \tilde{\mathcal{B}}_j$ with $V \cap U \neq \emptyset$. Indeed, if $U \cap V = \emptyset$ for each $V \in \bigcup_{j=0}^k \tilde{\mathcal{B}}_j$, then $U \in \mathcal{C}_k$ and $U \cap V = \emptyset$ for each $V \in \tilde{\mathcal{B}}_k$. Then, $\tilde{\mathcal{B}}_k \cup \{U\}$ would be a disjoint subcollection of \mathcal{C}_k , which contradicts the maximality of $\tilde{\mathcal{B}}_k$. So for $U \in \mathcal{B}_k$ there is $V \in \bigcup_{j=0}^k \tilde{\mathcal{B}}_j$ such that $U \cap V \neq \emptyset$. Then, $U \subseteq 5V$, where $5V$ denotes the ball with the same center as V but with 5 times its radius. Indeed, since $\text{diam } U \leq 2^{-k}R$ and $\text{diam } V > 2^{-(k+1)}R$, the triangle inequality yields the assertion. Summarizing we get

$$\lambda\left(\bigcup \mathcal{B}\right) = \lambda\left(\bigcup_{k \in \mathbb{N}} \bigcup \mathcal{B}_k\right) \leq \lambda\left(\bigcup_{k \in \mathbb{N}} \bigcup_{V \in \tilde{\mathcal{B}}_k} 5V\right) \leq 5^n \lambda\left(\bigcup \tilde{\mathcal{B}}\right). \quad \square$$

We will use Vitali's covering lemma to prove the weak 1-1 estimate for the Hardy-Littlewood maximal function, which is defined as follows.

Definition. Let X be a Banach space, $n \in \mathbb{N}_{>0}$, $f \in L_{1,\text{loc}}(\mathbb{R}^n; X)$. We denote the collection of all open and closed balls by $\mathcal{B} := \{B(y, r); y \in \mathbb{R}^n, r > 0\} \cup \{B[y, r]; y \in \mathbb{R}^n, r > 0\}$. Then the *Hardy-Littlewood maximal function* applied to f is given by

$$(Mf)(x) := \sup \left\{ \frac{1}{\lambda(B)} \int_B |f(y)| \, dy; B \in \mathcal{B}, x \in B \right\} \quad (x \in \mathbb{R}^n).$$

Lemma B.3. Let X be a Banach space, $n \in \mathbb{N}_{>0}$, $f \in L_{1,\text{loc}}(\mathbb{R}^n; X)$. Then Mf is lower-semicontinuous.

Proof. Let $t \geq 0$ and consider the set

$$U := \{x \in \mathbb{R}^n; (Mf)(x) > t\}.$$

Let $x \in U$. Then there exists $y \in \mathbb{R}^n, r > 0$ with $|x - y| \leq r$ such that

$$t < \frac{1}{\lambda(B(y, r))} \int_{B(y, r)} |f(z)| \, dz.$$

Choose now $r' > r$ such that

$$t < \frac{1}{\lambda(B(y, r'))} \int_{B(y, r')} |f(z)| \, dz.$$

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Then for each x' with $|x' - x| < r' - r$ we have $x' \in B(y, r')$ and thus,

$$t < \frac{1}{\lambda(B(y, r'))} \int_{B(y, r)} |f(z)| \, dz \leq \frac{1}{\lambda(B(y, r'))} \int_{B(y, r')} |f(z)| \, dz \leq (Mf)(x'),$$

hence $B(x, r' - r) \subseteq U$. Thus, U is open and hence, Mf is lower-semicontinuous. \square

Proposition B.4. *Let X be a Banach space, $n \in \mathbb{N}_{>0}$, $f \in L_1(\mathbb{R}^n; X)$ and $t > 0$. Then*

$$\lambda(\{x \in \mathbb{R}^n; (Mf)(x) > t\}) \leq \frac{5^n}{t} |f|_{L_1}.$$

Proof. Set $\mathcal{C} := \left\{ B \in \mathcal{B}; \frac{1}{\lambda(B)} \int_B |f| > t \right\}$. Then $\lambda(B) \leq \frac{|f|_1}{t}$ for each $B \in \mathcal{C}$, which in particular implies $\sup_{B \in \mathcal{C}} \text{diam } B < \infty$. By Proposition B.2 there exists a countable disjoint subcollection $\tilde{\mathcal{C}}$ of \mathcal{C} such that

$$\lambda\left(\bigcup \mathcal{C}\right) \leq 5^n \lambda\left(\bigcup \tilde{\mathcal{C}}\right).$$

Hence, we can estimate

$$\begin{aligned} \lambda(\{x \in \mathbb{R}^n; (Mf)(x) > t\}) &\leq \lambda\left(\bigcup \mathcal{C}\right) \\ &\leq 5^n \sum_{B \in \tilde{\mathcal{C}}} \lambda(B) \\ &\leq 5^n \sum_{B \in \tilde{\mathcal{C}}} \frac{1}{t} \int_B |f| \\ &\leq \frac{5^n}{t} |f|_{L_1}. \end{aligned} \quad \square$$

With the help of the latter estimate we are able to prove the Lebesgue Differentiation Theorem.

Theorem B.5 (Lebesgue Differentiation Theorem). *Let X be a Banach space, $n \in \mathbb{N}_{>0}$, $f \in L_{1,\text{loc}}(\mathbb{R}^n; X)$. For $x \in \mathbb{R}^n$ we define $\mathcal{B}_x := \{B \in \mathcal{B}; x \in B\}$. Then $(\mathcal{B}_x, \supseteq)$ is a directed set and for almost every $x \in \mathbb{R}^n$*

$$\lim_{B \in \mathcal{B}_x} \frac{1}{\lambda(B)} \int_B |f(y) - f(x)| \, dy = 0. \quad (\text{B.1})$$

Proof. As (B.1) is a local property, we may assume without loss of generality that $f \in L_1(\mathbb{R}^n; X)$. It suffices to prove that the set

$$M_\delta := \left\{ x \in \mathbb{R}^n; \limsup_{B \in \mathcal{B}_x} \frac{1}{\lambda(B)} \int_B |f(y) - f(x)| \, dy > \delta \right\}$$

has measure 0 for each $\delta > 0$. For doing so, let $\varepsilon, \delta > 0$ and choose $g \in C_c^\infty(\mathbb{R}^n; X)$ with

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$|f - g|_{L_1} < \varepsilon$. Then for $x \in \mathbb{R}^n$ we have

$$\begin{aligned} & \limsup_{B \in \mathcal{B}_x} \frac{1}{\lambda(B)} \int_B |f(y) - f(x)| \, dy \\ & \leq \limsup_{B \in \mathcal{B}_x} \frac{1}{\lambda(B)} \int_B |f(y) - g(y)| \, dy + \limsup_{B \in \mathcal{B}_x} \frac{1}{\lambda(B)} \int_B |g(y) - g(x)| \, dy + |g(x) - f(x)| \\ & \leq M(f - g)(x) + |g(x) - f(x)|, \end{aligned}$$

and thus,

$$\begin{aligned} \lambda(M_\delta) & \leq \lambda \left(\left\{ x \in \mathbb{R}^n; M(f - g)(x) > \frac{\delta}{2} \right\} \right) + \lambda \left(\left\{ x \in \mathbb{R}^n; |f(x) - g(x)| > \frac{\delta}{2} \right\} \right) \\ & \leq 5^n \frac{2}{\delta} |f - g|_{L_1} + \frac{2}{\delta} |f - g|_{L_1} \\ & < \frac{2(5^n + 1)}{\delta} \varepsilon, \end{aligned}$$

by Proposition B.4, which yields the assertion. \square

Using the latter theorem, we are able to prove Rademacher's Theorem in the scalar-valued case.

Theorem B.6 (Rademacher's Theorem; scalar-valued). *Let $F : I \subseteq \mathbb{R} \rightarrow \mathbb{K}$ Lipschitz-continuous, where I is an (possibly unbounded) interval and $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Then F is differentiable almost everywhere with $F' \in L_\infty(I)$ and $|F'|_{L_\infty} = |F|_{\text{Lip}}$. Moreover*

$$F(t) - F(s) = \int_s^t F' \, dx$$

for all $t, s \in I, s < t$.

Proof. We define the linear functional $\varphi : \text{lin} \{ \chi_{[s,t[} ; s, t \in I, s < t \} \subseteq L_1(I) \rightarrow \mathbb{K}$ by

$$\varphi \left(\sum_{i=1}^n \alpha_i \chi_{[s_i, t_i[} \right) := \sum_{i=1}^n \alpha_i (F(t_i) - F(s_i)),$$

for $n \in \mathbb{N}_{\geq 1}$, $\alpha_i \in \mathbb{K}$, $s_i, t_i \in I$, with $s_i < t_i$ for $i \in \{1, \dots, n\}$. Then, for pairwise disjoint intervals $[s_i, t_i[$, we have

$$\left| \varphi \left(\sum_{i=1}^n \alpha_i \chi_{[s_i, t_i[} \right) \right| \leq |F|_{\text{Lip}} \sum_{i=1}^n |\alpha_i| (t_i - s_i) = |F|_{\text{Lip}} \left| \sum_{i=1}^n \alpha_i \chi_{[s_i, t_i[} \right|_{L_1}$$

and thus, φ can be extended to an element in $L_1(I)'$, again denoted by φ , with $\|\varphi\| \leq |F|_{\text{Lip}}$. Since

$$|F(t) - F(s)| = |\varphi(\chi_{[s,t[})| \leq \|\varphi\| \|\chi_{[s,t[}\|_{L_1} = \|\varphi\|(t - s),$$

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for $s, t \in I, s < t$, we also obtain $|F|_{\text{Lip}} \leq \|\varphi\|$. Since $L_1(I)' \cong L_\infty(I)$, there exists $f \in L_\infty(I)$ with $|f|_{L_\infty} = \|\varphi\| = |F|_{\text{Lip}}$ such that

$$\varphi(g) = \int_I fg \quad (g \in L_1(I)).$$

In particular, we have

$$F(t) - F(s) = \int_s^t f$$

for $s, t \in I, s < t$ and hence, the assertion follows by Theorem B.5. \square

In order to prove Rademacher's Theorem for Banach space-valued functions, we need the following result due to Pettis ([Pet38]). We follow the proof given in [HP57, Theorem 3.5.3].

Theorem B.7 (Pettis). *Let X be a Banach space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, $(\Omega, \mathcal{F}, \mu)$ a σ -finite measure space and $f : \Omega \rightarrow X$. Then f is measurable¹ if and only if*

- (a) *f is weakly measurable, i.e. $x' \circ f : \Omega \rightarrow \mathbb{K}$ is measurable for each $x' \in X'$, and*
- (b) *f is almost separably valued, i.e. $\overline{\lim f[\Omega \setminus N]}$ is separable for some $N \in \mathcal{F}$ with $\mu(N) = 0$.*

Proof. If f is measurable, then clearly it is weakly measurable. Moreover, as f is the almost everywhere limit of simple functions, it is almost separably-valued. Assume now conversely that f satisfies (a) and (b). We define $Y := \overline{\lim f[\Omega \setminus N]}$, which is a separable Banach space by (b). Thus, there exists a sequence $(x'_n)_{n \in \mathbb{N}}$ in X' such that²

$$|y| = \sup_{n \in \mathbb{N}} |x'_n(y)| \quad (y \in Y).$$

Thus, by (a) we get that $(\Omega \setminus N \ni x \mapsto |f(x)| \in \mathbb{R})$ is measurable, as it is the supremum of a sequence of measurable functions. Hence, the set $F := \{x \in \Omega \setminus N; |f(x)| > 0\}$ is measurable. Let $\varepsilon > 0$, $(y_n)_{n \in \mathbb{N}}$ a dense sequence in Y and set

$$E_n := \{x \in F; |f(x) - y_n| < \varepsilon\} \quad (n \in \mathbb{N}).$$

Then E_n is measurable for each $n \in \mathbb{N}$ and $\bigcup_{n \in \mathbb{N}} E_n = F$ by the density of $\{y_n; n \in \mathbb{N}\}$. Setting $F_0 = E_0$ and $F_{n+1} = E_{n+1} \setminus \bigcup_{k=0}^n F_k$ for $n \in \mathbb{N}$, we obtain a sequence of pairwise disjoint measurable sets $(F_n)_{n \in \mathbb{N}}$ with $\bigcup_{n \in \mathbb{N}} F_n = F$. We set

$$g := \sum_{k=0}^{\infty} y_k \chi_{F_k}$$

and obtain $|f(x) - g(x)| < \varepsilon$ for each $x \in \Omega \setminus N$. Hence, if g is measurable, then so is f . For showing the measurability of g , let $(\Omega_k)_{k \in \mathbb{N}}$ be a sequence of pairwise disjoint measurable sets

¹Recall that f is called measurable, if there exists a sequence of simple functions, which converge to f almost everywhere. A simple function is a finitely-valued function $g : \Omega \rightarrow X$, such that $g^{-1}[\{x\}] \in \mathcal{F}$ with $\mu(g^{-1}[\{x\}]) < \infty$ for each $x \in X$.

²Choose a dense sequence $(y_n)_{n \in \mathbb{N}}$ in Y and let $x'_n \in X'$ with $\|x'_n\| = 1$ and $|x'_n(y_n)| = |y_n|$ for $n \in \mathbb{N}$.

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such that $\bigcup_{k \in \mathbb{N}} \Omega_k = \Omega$ and $\mu(\Omega_k) < \infty$ for each $k \in \mathbb{N}$. For $n \in \mathbb{N}$ we set

$$g_n := \sum_{k,j=0}^n y_k \chi_{F_k \cap \Omega_j}.$$

Then $(g_n)_{n \in \mathbb{N}}$ is a sequence of simple functions with $g_n \rightarrow g$ pointwise as $n \rightarrow \infty$ and thus, g is measurable. \square

We are now able to prove Rademacher's Theorem for Lipschitz-continuous functions with values in a reflexive Banach space. In fact, one can prove the Theorem for a larger class of Banach spaces, namely those having the Radon-Nikodym property, see e.g. [DU77, Theorem 2, p. 106]. More precisely, the validity of Rademacher's Theorem is equivalent to the fact that X satisfies the Radon-Nikodym property (cf. [Are87, Theorem 1.5]). But since we mainly deal with Hilbert spaces, we may restrict ourselves to this simpler situation.

Theorem B.8 (Rademacher's Theorem; Banach space-valued). *Let $F : I \subseteq \mathbb{R} \rightarrow X$ Lipschitz-continuous, where I is an (possibly unbounded) interval and X is a reflexive Banach space. Then F is differentiable almost everywhere with $F' \in L_\infty(I; X)$ and $|F'|_{L_\infty} = |F|_{\text{Lip}}$. Moreover*

$$F(t) - F(s) = \int_s^t F'$$

for all $t, s \in I, s < t$.

Proof. We consider the separable subspace $Y := \overline{\text{lin } F[I]}$, which is reflexive as it is a closed subspace of a reflexive space. For $y' \in Y'$ the function $y' \circ F : I \rightarrow \mathbb{K}$ is Lipschitz-continuous with $|y' \circ F|_{\text{Lip}} \leq \|y'\| \|F\|_{\text{Lip}}$ and hence, there exists a unique $f_{y'} \in L_\infty(I)$ with $|f_{y'}|_{L_\infty} \leq \|y'\| \|F\|_{\text{Lip}}$ and

$$y'(F(t) - F(s)) = \int_s^t f_{y'}$$

for each $s, t \in I$ with $s < t$ by Theorem B.6. Since Y is reflexive and separable, so is Y' . Let $A \subseteq S_{Y'}(0, 1)$ be a linear independent countable subset, which is dense in $S_{Y'}(0, 1)$ and set $Z := \text{lin}_{\mathbb{Q}} A$. Then Z is dense in Y' . For each $z' \in Z$ there exists a nullset $N_{z'}^{(1)} \subseteq I$ such that $|f_{z'}(x)| \leq |f_{z'}|_{L_\infty} \leq \|z'\| \|F\|_{\text{Lip}}$ for each $x \in I \setminus N_{z'}^{(1)}$. Moreover, for $z' = \sum_{i=1}^n \alpha_i y'_i$ with $\alpha_1, \dots, \alpha_n \in \mathbb{Q}, y'_1, \dots, y'_n \in A, n \in \mathbb{N}$ we have that

$$z'(F(t) - F(s)) = \int_s^t \sum_{i=1}^n \alpha_i f_{y'_i},$$

and hence, there exists a nullset $N_{z'}^{(2)} \subseteq I$ such that $f_{z'}(x) = \sum_{i=1}^n \alpha_i f_{y'_i}(x)$ for each $x \in I \setminus N_{z'}^{(2)}$. Thus, setting $N := \bigcup_{z' \in Z} N_{z'}^{(1)} \cup N_{z'}^{(2)}$, we infer that for each $x \in I \setminus N$ and $z'_1, z'_2 \in$

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$Z, \lambda \in \mathbb{Q}$ we have that

$$\begin{aligned} |f_{z'_1}(x)| &\leq \|z'_1\| \|F\|_{\text{Lip}}, \\ \lambda f_{z'_1}(x) + f_{z'_2}(x) &= f_{\lambda z'_1 + z'_2}(x). \end{aligned}$$

Hence, for each $x \in I \setminus N$ the mapping

$$\begin{aligned} \varphi_x : Z \subseteq Y' &\rightarrow \mathbb{K} \\ z' &\mapsto f_{z'}(x) \end{aligned}$$

is a linear bounded functional, which, due to the density of Z , can be extended to an element in $Y'' = Y$. We define the mapping

$$\begin{aligned} f : I &\rightarrow Y \\ x &\mapsto \begin{cases} \varphi_x & \text{if } x \in I \setminus N, \\ 0 & \text{else.} \end{cases} \end{aligned}$$

We first show that f is measurable. For doing so, we note that f is separably valued. Moreover, for each $y' \in Y'$ there is a sequence $(z'_n)_{n \in \mathbb{N}}$ in Z with $z'_n \rightarrow y'$ in Y' as $n \rightarrow \infty$. Thus, we have that

$$(y' \circ f)(x) = \lim_{n \rightarrow \infty} z'_n(f(x)) = \lim_{n \rightarrow \infty} \varphi_x(z'_n) = \lim_{n \rightarrow \infty} f_{z'_n}(x) \quad (x \in I \setminus N),$$

which implies the measurability of $y' \circ f$. Thus, by Theorem B.7 we derive the measurability of f . Moreover, we have that

$$|f(x)| = \sup_{z' \in A} |z'(f(x))| = \sup_{z' \in A} |f_{z'}(x)| \leq \|F\|_{\text{Lip}} \quad (x \in I \setminus N),$$

which shows that $f \in L_\infty(I; X)$ and $\|f\|_{L_\infty} \leq \|F\|_{\text{Lip}}$. Furthermore, for $z' \in Z$ we have

$$z'(F(t) - F(s)) = \int_s^t f_{z'}(x) \, dx = \int_s^t z'(f(x)) \, dx = z' \left(\int_s^t f(x) \, dx \right),$$

which yields

$$F(t) - F(s) = \int_s^t f(x) \, dx$$

for $s, t \in I$ with $s < t$. The latter implies $|F(t) - F(s)| \leq \|f\|_{L_\infty} |t - s|$, which shows $\|f\|_{L_\infty} = \|F\|_{\text{Lip}}$. That f is indeed the almost everywhere derivative of F follows from Theorem B.5. \square

C. The Widder-Arendt Theorem

In this part of the appendix we provide a proof of the Widder-Arendt Theorem. This theorem, which was proved by Widder for the scalar-valued case ([Wid34],[Wid71, Theorem 8, p. 157]) and generalized by Arendt to the Banach space-valued case ([Are87]), characterizes those functions, which are representable as the Laplace transform of a bounded measurable function. We mainly follow the rationale given in [ABHN11, Chapter 2].

Lemma C.1. *We define*

$$\varrho_{k,t}(s) := \frac{1}{k!} \left(\frac{k}{t} \right)^{k+1} e^{-\frac{k}{t}s} s^k \quad (s, t > 0, k \in \mathbb{N}_{\geq 1}).$$

Then for each $t > 0$ the sequence $(\varrho_{k,t})_{k \in \mathbb{N}_{\geq 1}}$ is a mollifier centered at t .

Proof. Let $t > 0$. Obviously, $\varrho_{k,t} \in C^\infty(\mathbb{R}_{>0})$ and $\varrho_{k,t} > 0$ for each k . Moreover, for $k \in \mathbb{N}_{\geq 1}$,

$$\begin{aligned} \int_0^\infty \varrho_{k,t}(s) \, ds &= \frac{1}{k!} \left(\frac{k}{t} \right)^{k+1} \int_0^\infty e^{-\frac{k}{t}s} s^k \, ds \\ &= \frac{k}{t} \int_0^\infty e^{-\frac{k}{t}s} \, ds \\ &= 1, \end{aligned}$$

by integration by parts. We define $g_k(s) := \frac{k^{k+1}}{k!} s^k e^{-ks} = t \varrho_{k,t}(st)$ for $k \in \mathbb{N}_{\geq 1}, s > 0$. Then

$$\begin{aligned} \int_0^{t-\varepsilon} \varrho_{k,t}(s) \, ds &= \int_0^{1-\frac{\varepsilon}{t}} g_k(s) \, ds, \\ \int_{t+\varepsilon}^\infty \varrho_{k,t}(s) \, ds &= \int_{1+\frac{\varepsilon}{t}}^\infty g_k(s) \, ds \end{aligned}$$

for each $k \in \mathbb{N}_{\geq 1}, \varepsilon > 0$. So, it suffices to prove that $\int_0^{1-\delta} g_k(s) \, ds \rightarrow 0$ ($k \rightarrow \infty$) and $\int_{1+\delta}^\infty g_k(s) \, ds \rightarrow 0$ ($k \rightarrow \infty$) for each $\delta > 0$. For doing so, we define $h(s) := s e^{1-s}$ for $s > 0$. Then, since $\frac{k^k}{k!} < e^k$, we obtain $g_k(s) \leq k(h(s))^k$ for each $k \in \mathbb{N}_{\geq 1}, s > 0$. We observe that h strictly increases on $]0, 1[$. Thus, we derive

$$\int_0^{1-\delta} g_k(s) \, ds \leq k \int_0^{1-\delta} h(1-\delta)^k \, ds \leq kh(1-\delta)^k(1-\delta) \rightarrow 0 \quad (k \rightarrow \infty)$$

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since $h(1-\delta) < h(1) = 1$. Moreover, we observe that $h(s)e^{as} \leq \frac{1}{1-a}$ for each $s > 0, 0 < a < 1$. Hence,

$$\begin{aligned} \int_{1+\delta}^{\infty} g_k(s) \, ds &\leq k \left(\frac{1}{1-a} \right)^k \int_{1+\delta}^{\infty} e^{-aks} \, ds \\ &= \left(\frac{1}{1-a} \right)^k \frac{1}{a} e^{-ak(1+\delta)}. \end{aligned}$$

Choosing now $a := \frac{\delta}{1+\delta}$, we derive

$$\int_{1+\delta}^{\infty} g_k(s) \, ds \leq \frac{(1+\delta)^{k+1}}{\delta} e^{-k\delta} \rightarrow 0 \quad (k \rightarrow \infty). \quad \square$$

Next we state the Post-Widder inversion formula for bounded continuous functions. We note that this result also holds for locally integrable functions (see e.g. [ABHN11, Theorem 1.7.7]), however, as the proof for the more abstract result is rather technical, we stick to the simpler case.

Lemma C.2 (Post-Widder Inversion Formula). *Let X be a Banach space and $f \in C_b(\mathbb{R}_{\geq 0}; X)$. Then, for each $t > 0$ we have*

$$f(t) = \lim_{k \rightarrow \infty} (-1)^k \frac{1}{k!} \left(\frac{k}{t} \right)^{k+1} \sqrt{2\pi} \widehat{f}^{(k)} \left(\frac{k}{t} \right).$$

Proof. We have that

$$\widehat{f}^{(k)}(z) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-zs} (-s)^k f(s) \, ds \quad (k \in \mathbb{N}, z \in \mathbb{C}_{\text{Re}>0}).$$

In particular, we get for $k \in \mathbb{N}_{\geq 1}$ and $t > 0$

$$(-1)^k \frac{1}{k!} \left(\frac{k}{t} \right)^{k+1} \sqrt{2\pi} \widehat{f}^{(k)} \left(\frac{k}{t} \right) = \int_0^{\infty} (-1)^k \frac{1}{k!} \left(\frac{k}{t} \right)^{k+1} e^{-\frac{k}{t}s} (-s)^k f(s) \, ds = \int_0^{\infty} \varrho_{k,t}(s) f(s) \, ds,$$

where $\varrho_{k,t}$ is given as in Lemma C.1. For $\varepsilon > 0, t > 0$ we choose $\delta > 0$ such that $|f(s) - f(t)| < \varepsilon$ for $|s - t| < \delta$. Then, we obtain

$$\begin{aligned} \left| (-1)^k \frac{1}{k!} \left(\frac{k}{t} \right)^{k+1} \sqrt{2\pi} \widehat{f}^{(k)} \left(\frac{k}{t} \right) - f(t) \right| &\leq \int_0^{\infty} \varrho_{k,t}(s) |f(s) - f(t)| \, ds \\ &\leq 2|f|_{\infty} \left(\int_0^{t-\delta} \varrho_{k,t}(s) \, ds + \int_{t+\delta}^{\infty} \varrho_{k,t}(s) \, ds \right) + \varepsilon \\ &\leq 2\varepsilon \end{aligned}$$

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for sufficiently large k according to Lemma C.1. \square

Theorem C.3 (Riesz-Stieltjes representation). *Let X be a Banach space and $T : L_1(\mathbb{R}_{\geq 0}) \rightarrow X$ linear and bounded. We define*

$$F(t) = T\chi_{[0,t[} \quad (t \geq 0).$$

Then $F : \mathbb{R}_{\geq 0} \rightarrow X$ is Lipschitz-continuous with $F(0) = 0$ and $|F|_{\text{Lip}} = \|T\|$ and for each continuous $g \in L_1(\mathbb{R}_{\geq 0})$, we have¹

$$Tg = \lim_{t \rightarrow \infty} \int_0^t g(s) \, dF(s).$$

Proof. We have

$$|F(t) - F(s)| = |T(\chi_{[0,t[} - \chi_{[0,s[})| \leq \|T\||t - s| \quad (t, s \geq 0),$$

and hence, F is Lipschitz-continuous with $|F|_{\text{Lip}} \leq \|T\|$. For $g = \sum_{j=1}^n c_j \chi_{[\alpha_j, \alpha_{j+1}[}$, where $c_1, \dots, c_n \in \mathbb{R}$ and $0 \leq \alpha_1 < \dots < \alpha_n$ we have that

$$\begin{aligned} Tg &= \sum_{j=1}^n c_j T\chi_{[\alpha_j, \alpha_{j+1}[} \\ &= \sum_{j=1}^n c_j (T\chi_{[0, \alpha_{j+1}[} - T\chi_{[0, \alpha_j[}) \\ &= \sum_{j=1}^n c_j (F(\alpha_{j+1}) - F(\alpha_j)) \\ &= \int_0^\infty g(s) \, dF(s). \end{aligned}$$

In particular, $|Tg| \leq |F|_{\text{Lip}} \sum_{j=1}^n c_j |\alpha_{j+1} - \alpha_j| = |F|_{\text{Lip}} |g|_{L_1}$, and since simple functions are dense in $L_1(\mathbb{R}_{\geq 0})$, we derive $\|T\| \leq |F|_{\text{Lip}}$. Let now $g \in L_1(\mathbb{R}_{\geq 0})$ continuous and $t > 0$. Then there exists a sequence of simple functions $(g_n)_{n \in \mathbb{N}}$ such that $g_n|_{[0,t]} \rightarrow g|_{[0,t]}$ in $L_1(\mathbb{R})$ and point-wise. Thus,

$$\int_0^t g(s) \, dF(s) = \lim_{n \rightarrow \infty} \int_0^t g_n(s) \, dF(s) = \lim_{n \rightarrow \infty} T(g_n|_{[0,t]}) = T(g|_{[0,t]}).$$

Moreover, since $g|_{[0,t]} \rightarrow g$ in $L_1(\mathbb{R}_{\geq 0})$ as $t \rightarrow \infty$ we obtain the last assertion. \square

Before we come to our main result, we need the following technical lemma.

¹We note that on a compact interval, we have that F is of bounded variation and hence, the integral $\int_0^t g(s) \, dF(s)$ exists for continuous functions g .

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Lemma C.4. *Let X be a Banach space, $r \in C^\infty(\mathbb{R}_{>0}; X)$ such that*

$$M := \sup_{k \in \mathbb{N}, \lambda > 0} \left| \frac{r^{(k)}(\lambda) \lambda^{k+1}}{k!} \right| < \infty.$$

Then, for $\lambda > 0$ we have

$$\int_0^\infty e^{-\lambda t} (-1)^k \frac{1}{k!} \left(\frac{k}{t} \right)^{k+1} r^{(k)} \left(\frac{k}{t} \right) dt \rightarrow r(\lambda) \quad (k \rightarrow \infty).$$

Proof. Let $\lambda > 0$. We first prove that the integrals exist. For doing so, we estimate

$$\int_0^\infty \left| e^{-\lambda t} (-1)^k \frac{1}{k!} \left(\frac{k}{t} \right)^{k+1} r^{(k)} \left(\frac{k}{t} \right) \right| dt \leq M \int_0^\infty e^{-\lambda t} dt < \infty,$$

since $\lambda > 0$. We compute

$$\begin{aligned} \int_0^\infty e^{-\lambda t} (-1)^k \frac{1}{k!} \left(\frac{k}{t} \right)^{k+1} r^{(k)} \left(\frac{k}{t} \right) dt &= (-1)^k \frac{1}{(k-1)!} \int_0^\infty e^{-\lambda \frac{k}{s}} s^{k-1} r^{(k)}(s) ds \\ &= (-1)^k \frac{(\lambda k)^{k-1}}{(k-1)!} \int_0^\infty G_k(\lambda k, s) r^{(k)}(s) ds, \end{aligned}$$

where $G_k(x, s) := e^{-\frac{x}{s}} \left(\frac{s}{x} \right)^{k-1}$ for $x, s > 0$. Integration by parts then yields

$$\begin{aligned} &\int_0^\infty e^{-\lambda t} (-1)^k \frac{1}{k!} \left(\frac{k}{t} \right)^{k+1} r^{(k)} \left(\frac{k}{t} \right) dt \\ &= \frac{(-1)^k (\lambda k)^{k-1}}{(k-1)!} \left(\sum_{j=0}^{k-1} (-1)^j \left(\partial_2^j G_k \right) (\lambda k, s) r^{(k-j-1)}(s) \Big|_{s=0}^\infty + (-1)^k \int_0^\infty \left(\partial_2^k G_k \right) (\lambda k, s) r(s) ds \right). \end{aligned}$$

We want to compute the derivatives $\partial_2^j G_k$. First, we observe that $G_k(tx, ts) = G_k(x, s)$ for each $x, s, t > 0$. Hence,

$$0 = \partial(t \mapsto G_k(tx, ts))(1) = x (\partial_1 G_k)(x, s) + s (\partial_2 G_k)(x, s)$$

or equivalently

$$\frac{1}{x} (\partial_2 G_k)(x, s) = -\frac{1}{s} (\partial_1 G_k)(x, s)$$

for each $x, s > 0$. We now prove by induction, that

$$\frac{1}{x} \left(\partial_2^j G_k \right) (x, s) = \frac{1}{s} (-1)^j \left(\partial_1^j G_{k-j+1} \right) (x, s) \quad (x, s > 0, j \in \{1, \dots, k\}).$$

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The case $j = 1$ we have proved above. Assume now that the formula holds for j . Then

$$\begin{aligned}
& \frac{1}{x} \left(\partial_2^{j+1} G_k \right) (x, s) \\
&= \partial_2 \left(\frac{1}{m_2} (-1)^j \partial_1^j G_{k-j+1} \right) (x, s) \\
&= (-1)^{j+1} \frac{1}{s^2} \left(\partial_1^j G_{k-j+1} \right) (x, s) + \frac{1}{s} (-1)^j \left(\partial_1^j \partial_2 G_{k-j+1} \right) (x, s) \\
&= \frac{1}{s} (-1)^{j+1} \left(\frac{1}{s} \left(\partial_1^j G_{k-j+1} \right) (x, s) - \left(\partial_1^j \partial_2 G_{k-j+1} \right) (x, s) \right) \\
&= \frac{1}{s} (-1)^{j+1} \left(\frac{1}{s} \left(\partial_1^j G_{k-j+1} \right) (x, s) + \left(\partial_1^j \left(\frac{m_1}{m_2} \partial_1 G_{k-j+1} \right) \right) (x, s) \right) \\
&= \frac{1}{s} (-1)^{j+1} \left(\frac{1}{s} \partial_1^j G_{k-j+1} + \partial_1^{j+1} \left(\frac{m_1}{m_2} G_{k-j+1} \right) - \partial_1^j \left(\frac{1}{m_2} G_{k-j+1} \right) \right) (x, s) \\
&= \frac{1}{s} (-1)^{j+1} \left(\partial_1^{j+1} \left(\frac{m_1}{m_2} G_{k-j+1} \right) \right) (x, s) \\
&= \frac{1}{s} (-1)^{j+1} \left(\partial_1^{j+1} G_{k-j} \right) (x, s),
\end{aligned}$$

which proves the assertion. From this we derive

$$\begin{aligned}
& \left| \sum_{j=0}^{k-1} (-1)^j \left(\partial_2^j G_k \right) (\lambda k, s) r^{(k-j-1)}(s) \right| \\
&= \left| \sum_{j=0}^{k-1} \left(\partial_1^j G_{k-j+1} \right) (\lambda k, s) \frac{\lambda k}{s} r^{(k-j-1)}(s) \right| \\
&= (\lambda k) \left| \sum_{j=0}^{k-1} \partial^j \left(x \mapsto e^{-\frac{x}{s}} \left(\frac{1}{x} \right)^{k-j} \right) (\lambda k) s^{k-j-1} r^{(k-j-1)}(s) \right| \\
&\leq (\lambda k) M \sum_{j=0}^{k-1} \left| \frac{1}{s} \partial^j \left(x \mapsto e^{-\frac{x}{s}} x^{-k+j} \right) (\lambda k) (k-j-1)! \right|.
\end{aligned}$$

Since $\left| \partial^j \left(x \mapsto e^{-\frac{x}{s}} x^{-k+j} \right) (\lambda k) \right| \leq C |p(s^{-1})| e^{-\frac{\lambda k}{s}}$ for some constant C and some polynomial p we get

$$\sum_{j=0}^{k-1} (-1)^j \left(\partial_2^j G_k \right) (\lambda k, s) r^{(k-j-1)}(s) \Big|_{s=0}^\infty = 0.$$

Thus, we have that

$$\begin{aligned}
\int_0^\infty e^{-\lambda t} (-1)^k \frac{1}{k!} \left(\frac{k}{t} \right)^{k+1} r^{(k)} \left(\frac{k}{t} \right) dt &= \frac{(\lambda k)^{k-1}}{(k-1)!} \int_0^\infty \left(\partial_2^k G_k \right) (\lambda k, s) r(s) ds \\
&= \frac{(\lambda k)^k}{(k-1)!} \int_0^\infty (-1)^k \frac{1}{s} \left(\partial_1^k G_1 \right) (\lambda k, s) r(s) ds
\end{aligned}$$

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$$\begin{aligned}
&= \frac{(\lambda k)^k}{(k-1)!} \int_0^\infty \frac{1}{s^{k+1}} e^{-\frac{\lambda k}{s}} r(s) \, ds \\
&= \frac{(\lambda k)^k}{(k-1)!} \int_0^\infty t^{k-1} e^{-\lambda k t} r\left(\frac{1}{t}\right) \, dt \\
&= \frac{(\lambda k)^k}{(k-1)!} \int_0^\infty t^k e^{-\lambda k t} f(t) \, dt \\
&= \frac{(\lambda k)^k}{(k-1)!} \sqrt{2\pi} (-1)^k \widehat{f}^{(k)}(\lambda k),
\end{aligned}$$

where $f(t) := \frac{1}{t} r\left(\frac{1}{t}\right)$ for $t > 0$. Since $f \in C_b(\mathbb{R}_{>0}; X)$ we obtain by Lemma C.2 (choose $t = \frac{1}{\lambda}$)

$$\frac{(\lambda k)^k}{(k-1)!} \sqrt{2\pi} (-1)^k \widehat{f}^{(k)}(\lambda k) = \frac{1}{\lambda} \frac{(\lambda k)^{k+1}}{k!} \sqrt{2\pi} (-1)^k \widehat{f}^{(k)}(\lambda k) \rightarrow \frac{1}{\lambda} f\left(\frac{1}{\lambda}\right) = r(\lambda) \quad (k \rightarrow \infty).$$

□

Theorem C.5. *Let X be a Banach space, $r \in C^\infty(\mathbb{R}_{>0}; X)$ such that*

$$M := \sup_{\lambda > 0, k \in \mathbb{N}} \left| \frac{r^{(k)}(\lambda) \lambda^{k+1}}{k!} \right| < \infty.$$

Then there exists $F : \mathbb{R}_{\geq 0} \rightarrow X$ Lipschitz-continuous with $F(0) = 0$ such that

$$r(\lambda) = \int_0^\infty e^{-\lambda t} \, dF(t) \quad (\lambda > 0).$$

Moreover $|F|_{\text{Lip}} = M$.

Proof. For $k \in \mathbb{N}$ we define the operator $T_k : L_1(\mathbb{R}_{\geq 0}) \rightarrow X$ by

$$T_k f := \int_0^\infty f(t) (-1)^k \frac{1}{k!} \left(\frac{k}{t}\right)^{k+1} r^{(k)}\left(\frac{k}{t}\right) \, dt \quad (f \in L_1(\mathbb{R}_{\geq 0})).$$

Then T_k is linear and since $|T_k f| \leq M |f|_{L_1}$ for $f \in L_1(\mathbb{R}_{\geq 0})$, it is also bounded. By Lemma C.4 we have

$$T_k e^{-\lambda \cdot} \rightarrow r(\lambda) \quad (k \rightarrow \infty)$$

for each $\lambda > 0$. As $\{e^{-\lambda \cdot} ; \lambda > 0\}$ is a total set in $L_1(\mathbb{R}_{\geq 0})$ (see Corollary A.9) and $\sup_{k \in \mathbb{N}} \|T_k\| \leq M$, we derive that there exists a bounded linear operator $T : L_1(\mathbb{R}_{\geq 0}) \rightarrow X$ with $\|T\| \leq M$ and

$$T_k f \rightarrow T f \quad (k \rightarrow \infty)$$

for each $f \in L_1(\mathbb{R}_{\geq 0})$. In particular,

$$T e^{-\lambda \cdot} = r(\lambda)$$

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for each $\lambda > 0$. Applying now Theorem C.3, we find $F : \mathbb{R}_{\geq 0} \rightarrow X$ Lipschitz with $F(0) = 0$, $|F|_{\text{Lip}} = \|T\| \leq M$ and

$$\int_0^\infty e^{-\lambda t} dF(t) = T e^{-\lambda \cdot} = r(\lambda) \quad (\lambda > 0).$$

It is left to show $M \leq |F|_{\text{Lip}}$. We observe that

$$\begin{aligned} |r^{(k)}(\lambda)| &= \left| \int_0^\infty e^{-\lambda t} (-t)^k dF(t) \right| \\ &\leq \|T\| \int_0^\infty t^k e^{-\lambda t} dt \\ &= |F|_{\text{Lip}} k! \frac{1}{\lambda^{k+1}} \quad (k \in \mathbb{N}, \lambda > 0) \end{aligned}$$

and thus, $M = \sup_{\lambda > 0, k \in \mathbb{N}} \left| \frac{r^{(k)}(\lambda) \lambda^{k+1}}{k!} \right| \leq |F|_{\text{Lip}}$. \square

Using Rademacher's Theorem (Theorem B.8), we are now able to prove the Theorem of Widder-Arendt. Again we just state the theorem for reflexive Banach spaces and note that it also holds for Banach spaces with the Radon-Nikodym property (in fact, it is even equivalent to X having the Radon-Nikodym property, see [Are87, Theorem 1.4]).

Theorem C.6 (Widder-Arendt). *Let X be a reflexive Banach space and $r \in C^\infty(\mathbb{R}_{>0}; X)$ such that*

$$M := \sup_{\lambda > 0, k \in \mathbb{N}} \left| \frac{r^{(k)}(\lambda) \lambda^{k+1}}{k!} \right| < \infty.$$

Then there exists $f \in L_\infty(\mathbb{R}_{>0}; X)$ such that $|f|_\infty = M$ and

$$r(\lambda) = \int_0^\infty e^{-\lambda t} f(t) dt.$$

Proof. By Theorem C.5 there is $F : \mathbb{R}_{\geq 0} \rightarrow X$ Lipschitz continuous with $F(0) = 0$, $|F|_{\text{Lip}} = M$ and

$$r(\lambda) = \int_0^\infty e^{-\lambda t} dF(t).$$

By Theorem B.8, F is differentiable almost everywhere with a bounded derivative $f := F'$ with $|f|_\infty = |F|_{\text{Lip}} = M$. Moreover,

$$F(t) - F(s) = \int_s^t f = \int_0^\infty \chi_{[s,t]} f$$

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for each $0 \leq s < t$. By continuous extension we get

$$\int_0^\infty g \, dF = \int_0^\infty gf$$

for each continuous function $g \in L_1(\mathbb{R}_{>0}; X)$ and thus, especially

$$r(\lambda) = \int_0^\infty e^{-\lambda t} \, dF(t) = \int_0^\infty e^{-\lambda t} f(t) \, dt \quad (\lambda > 0). \quad \square$$

Bibliography

- [AB88] W. Arendt and C. Batty. Tauberian theorems and stability of one-parameter semigroups. *Trans. Am. Math. Soc.*, 306(2):837–852, 1988.
- [ABHN11] W. Arendt, C. J. Batty, M. Hieber, and F. Neubrander. *Vector-valued Laplace transforms and Cauchy problems. 2nd ed.* Basel: Birkhäuser, 2011.
- [Are87] W. Arendt. Vector-valued Laplace transforms and Cauchy problems. *Isr. J. Math.*, 59:327–352, 1987.
- [AV10] A. Alonso Rodríguez and A. Valli. *Eddy current approximation of Maxwell equations. Theory, algorithms and applications.* Milano: Springer, 2010.
- [BD82] A. Bukhvalov and A. Danilevich. Boundary properties of analytic and harmonic functions with values in Banach space. *Math. Notes*, 31:104–110, 1982.
- [BD08] C. J. Batty and T. Duyckaerts. Non-uniform stability for bounded semi-groups on Banach spaces. *J. Evol. Equ.*, 8(4):765–780, 2008.
- [Ber86] I. U. M. Berezanskii. *Selfadjoint operators in spaces of functions of infinitely many variables.* Translations of mathematical monographs. American Mathematical Society, 1986.
- [BH76] H. Brézis and A. Haraux. Image d’une somme d’opérateurs monotones et applications. *Isr. J. Math.*, 23:165–186, 1976.
- [BP01] A. Bátkai and S. Piazzera. Semigroups and linear partial differential equations with delay. *J. Math. Anal. Appl.*, 264(1):1–20, 2001.
- [BP05] A. Bátkai and S. Piazzera. *Semigroups for delay equations.* Research Notes in Mathematics 10. Wellesley, MA: A K Peters. xii, 2005.
- [BQR14] K. Borgmeyer, R. Quintanilla, and R. Racke. Phase-lag heat conduction: decay rates for limit problems and well-posedness. *J. Evol. Equ.*, 14(4-5):863–884, 2014.
- [Bre71] H. Brezis. *Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert.* Université Paris VI et CNRS, 1971.
- [Bre11] H. Brezis. *Functional analysis, Sobolev spaces and partial differential equations.* New York, NY: Springer, 2011.
- [BT10] A. Borichev and Y. Tomilov. Optimal polynomial decay of functions and operator semigroups. *Math. Ann.*, 347(2):455–478, 2010.
- [Buk81] A. Bukhvalov. Hardy spaces of vector-valued functions. *J. Sov. Math.*, 16:1051–1059, 1981.

Bibliography

- [CF95] P. Colli and A. Favini. On some degenerate second order equations of mixed type. *Funkc. Ekvacioj, Ser. Int.*, 38(3):473–489, 1995.
- [CF96] P. Colli and A. Favini. Time discretization of nonlinear Cauchy problems applying to mixed hyperbolic-parabolic equations. *Int. J. Math. Math. Sci.*, 19(3):481–494, 1996.
- [CG82] G. Chen and R. Grimmer. Integral equations as evolution equations. *J. Differ. Equations*, 45:53–74, 1982.
- [CS03] P. Cannarsa and D. Sforza. Global solutions of abstract semilinear parabolic equations with memory terms. *NoDEA, Nonlinear Differ. Equ. Appl.*, 10(4):399–430, 2003.
- [CS08] P. Cannarsa and D. Sforza. A stability result for a class of nonlinear integrodifferential equations with L^1 kernels. *Appl. Math.*, 35(4):395–430, 2008.
- [CS11] P. Cannarsa and D. Sforza. Integro-differential equations of hyperbolic type with positive definite kernels. *J. Differ. Equations*, 250(12):4289–4335, 2011.
- [Daf70a] C. Dafermos. An abstract Volterra equation with applications to linear viscoelasticity. *J. Differ. Equations*, 7:554–569, 1970.
- [Daf70b] C. Dafermos. Asymptotic stability in viscoelasticity. *Arch. Ration. Mech. Anal.*, 37:297–308, 1970.
- [Dat70] R. Datko. Extending a theorem of A. M. Liapunov to Hilbert space. *J. Math. Anal. Appl.*, 32:610–616, 1970.
- [Dat72] R. Datko. Uniform asymptotic stability of evolutionary processes in a Banach space. *SIAM J. Math. Anal.*, 3:428–445, 1972.
- [deS64] L. de Simon. Un’applicazione della teoria degli integrali singolari allo studio delle equazioni differenziali lineari astratte del primo ordine. *Rend. Semin. Mat. Univ. Padova*, 34:205–223, 1964.
- [dPG75] G. da Prato and P. Grisvard. Sommes d’opérateurs linéaires et équations différentielles opérationnelles. *J. Math. Pures Appl. (9)*, 54:305–387, 1975.
- [DU77] J. Diestel and J. Uhl. Vector measures. Mathematical Surveys. No.15. Providence, R.I.: American Mathematical Society (AMS). XIII., 1977.
- [DV87] G. Dore and A. Venni. On the closedness of the sum of two closed operators. *Math. Z.*, 196:189–201, 1987.
- [EN00] K. J. Engel and R. Nagel. *One-parameter semigroups for linear evolution equations*. Graduate texts in mathematics. Springer, 2000.
- [Eng98] K.-J. Engel. On the characterization of admissible control- and observation operators. *Syst. Control Lett.*, 34(4):225–227, 1998.
- [Eva10] L. C. Evans. *Partial differential equations*. Graduate studies in mathematics. American Mathematical Society, 2010.
- [Fri58] K. Friedrichs. Symmetric positive linear differential equations. *Commun. Pure Appl. Math.*, 11:333–418, 1958.

Bibliography

- [FS55] Y. Fourès and I. Segal. Causality and analyticity. *Trans. Am. Math. Soc.*, 78:385–405, 1955.
- [FY99] A. Favini and A. Yagi. *Degenerate differential equations in Banach spaces*. New York, NY: Marcel Dekker, 1999.
- [Gea78] L. Gearhart. Spectral theory for contraction semigroups on Hilbert space. *Trans. Am. Math. Soc.*, 236:385–394, 1978.
- [GLS90] G. Gripenberg, S.-O. Londen, and O. Staffans. *Volterra integral and functional equations*. Cambridge etc.: Cambridge University Press, 1990.
- [GVW81] G. Greiner, J. Voigt, and M. Wolff. On the spectral bound of the generator of semigroups of positive operators. *J. Oper. Theory*, 5:245–256, 1981.
- [Hal71] J. Hale. Functional differential equations. Applied Mathematical Sciences. Vol. 3. New York-Heidelberg-Berlin: Springer-Verlag. VIII, 1971.
- [Her83] I. Herbst. The spectrum of Hilbert space semigroups. *J. Oper. Theory*, 10:87–94, 1983.
- [HL93] J. Hale and S. Lunel. *Introduction to Functional Differential Equations*. Applied Mathematical Sciences. Springer, 1993.
- [HP57] E. Hille and R. S. Phillips. *Functional analysis and semi-groups*. Colloquium Publications - American Mathematical Society. American Mathematical Society, 1957.
- [HP97] S. Hu and N. S. Papageorgiou. *Handbook of Multivalued Analysis*, volume 1: Theory. Springer, 1997.
- [KL00] N. Kalton and G. Lancien. A solution to the problem of L^p -maximal regularity. *Math. Z.*, 235(3):559–568, 2000.
- [KM06] P. Kunkel and V. Mehrmann. *Differential-algebraic equations. Analysis and numerical solution*. Zürich: European Mathematical Society Publishing House, 2006.
- [KPR15] D. Khusainov, M. Pokojovy, and R. Racke. Strong and mild extrapolated L^2 -solutions to the heat equation with constant delay. *SIAM J. Math. Anal.*, 47(1):427–454, 2015.
- [KPS⁺14] A. Kalauch, R. Picard, S. Siegmund, S. Trostorff, and M. Waurick. A Hilbert space perspective on ordinary differential equations with memory term. *J. Dyn. Differ. Equations*, 26(2):369–399, 2014.
- [KS83] K. Kunisch and W. Schappacher. Necessary conditions for partial differential equations with delay to generate C_0 -semigroups. *J. Differ. Equations*, 50:49–79, 1983.
- [Kwa72] S. Kwapień. Isomorphic characterizations of inner product spaces by orthogonal series with vector valued coefficients. *Studia Mathematica*, 44(6):583–595, 1972.
- [LLM10] V. Lakshmikantham, S. Leela, and F. A. Mcrae. *Theory of Causal Differential Equations*. Atlantis Studies in Mathematics for Engineering and Science. World Scientific Pub Co Inc, 2010.

Bibliography

- [LV88] Y. Lyubich and Vu Quốc Phóng. Asymptotic stability of linear differential equations in Banach spaces. *Stud. Math.*, 88(1):37–42, 1988.
- [MC96] M. A. Murad and J. H. Cushman. Multiscale flow and deformation in hydrophilic swelling porous media. *Int. J. Eng. Sci.*, 34(3):313–338, 1996.
- [Min62] G. Minty. Monotone (nonlinear) operators in a hilbert space. *Duke Math. J.*, 29, 1962.
- [Mor52] D. Morgenstern. *Beiträge zur nichtlinearen Funktionalanalysis*. PhD thesis, TU Berlin, 1952.
- [MP10] D. F. McGhee and R. Picard. A note on anisotropic, inhomogeneous, poro-elastic media. *Math. Methods Appl. Sci.*, 33(3):313–322, 2010.
- [MPTW15] S. Mukhopadhyay, R. Picard, S. Trostorff, and M. Waurick. A note on a two-temperature model in linear thermoelasticity. *Math. Mech. Solids*, 2015. doi:10.1177/1081286515611947.
- [MPTW16a] S. Mukhopadhyay, R. Picard, S. Trostorff, and M. Waurick. On some models in linear thermo-elasticity with rational material laws. *Math. Mech. Solids*, 21(9):1149–1163, 2016.
- [MPTW16b] A. J. Mulholland, R. Picard, S. Trostorff, and M. Waurick. On well-posedness for some thermo-piezoelectric coupling models. *Math. Methods Appl. Sci.*, 39(15):4375–4384, 2016.
- [Nag97] R. Nagel. Extrapolation spaces for semigroups. *RIMS Kokyuroku*, 1009:181–191, 1997.
- [Neu86] F. Neubrander. Laplace transform and asymptotic behavior of strongly continuous semigroups. *Houston J. Math.*, 12:549–561, 1986.
- [NS76] J. Nohel and D. Shea. Frequency domain methods for Volterra equations. *Adv. Math.*, 22:278–304, 1976.
- [Paz83] A. Pazy. Semigroups of linear operators and applications to partial differential equations. Applied Mathematical Sciences, 44. New York etc.: Springer-Verlag. VIII, 1983.
- [Pet38] B. J. Pettis. On integration in vector spaces. *Trans. Am. Math. Soc.*, 44:277–304, 1938.
- [Phi59] R. Phillips. Dissipative operators and hyperbolic systems of partial differential equation. *Trans. Am. Math. Soc.*, 90:193–254, 1959.
- [Pic89] R. H. Picard. *Hilbert space approach to some classical transforms*. Pitman research notes in mathematics series. Longman Scientific & Technical, 1989.
- [Pic00] R. Picard. Evolution equations as operator equations in lattices of Hilbert spaces. *Glas. Mat., III. Ser.*, 35(1):111–136, 2000.
- [Pic09] R. Picard. A structural observation for linear material laws in classical mathematical physics. *Math. Methods Appl. Sci.*, 32(14):1768–1803, 2009.
- [Pic12] R. Picard. A class of evolutionary problems with an application to acoustic waves with impedance type boundary conditions. In *Spectral Theory*,

Bibliography

- Mathematical System Theory, Evolution Equations, Differential and Difference Equations*, volume 221 of *Operator Theory: Advances and Applications*, pages 533–548. Springer Basel, 2012.
- [PM11] R. Picard and D. McGhee. *Partial differential equations. A unified Hilbert space approach*. de Gruyter Expositions in Mathematics 55. Berlin: de Gruyter. xviii, 2011.
- [Pod99] I. Podlubny. *Fractional differential equations. An introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications*. San Diego, CA: Academic Press, 1999.
- [PP16] D. Pauly and R. Picard. A note on the justification of the eddy current model in electrodynamics. Technical report, 2016. arXiv: 1606.00472.
- [Prü84] J. Prüss. On the spectrum of C_0 -semigroups. *Trans. Am. Math. Soc.*, 284:847–857, 1984.
- [Prü93] J. Prüss. *Evolutionary integral equations and applications*. Monographs in mathematics. Birkhäuser Verlag, 1993.
- [Prü09] J. Prüss. Decay properties for the solutions of a partial differential equation with memory. *Arch. Math.*, 92(2):158–173, 2009.
- [PSC02] T. Pratchett, I. Stewart, and J. Cohen. *The Science of Discworld II: The Globe*. Random House, 2002.
- [PSTW16] R. Picard, S. Seidler, S. Trostorff, and M. Waurick. On abstract grad-div systems. *J. Differ. Equations*, 260(6):4888–4917, 2016.
- [PTW13] R. Picard, S. Trostorff, and M. Waurick. A note on a class of conservative, well-posed linear control systems. In M. Reissig and M. Ruzhansky, editors, *Progress in Partial Differential Equations*, volume 44 of *Springer Proceedings in Mathematics & Statistics*, pages 261–286. Springer International Publishing, 2013.
- [PTW14a] R. Picard, S. Trostorff, and M. Waurick. A functional analytic perspective to delay differential equations. *Oper. Matrices*, 8(1):217–236, 2014.
- [PTW14b] R. Picard, S. Trostorff, and M. Waurick. On a class of boundary control problems. *Oper. Matrices*, 8(1):185–204, 2014.
- [PTW15a] R. Picard, S. Trostorff, and M. Waurick. On evolutionary equations with material laws containing fractional integrals. *Math. Methods Appl. Sci.*, 38(15):3141–3154, 2015.
- [PTW15b] R. Picard, S. Trostorff, and M. Waurick. On some models for elastic solids with micro-structure. *ZAMM, Z. Angew. Math. Mech.*, 95(7):664–689, 2015.
- [PTW16a] R. Picard, S. Trostorff, and M. Waurick. On a comprehensive Class of Linear Control Problems. *IMA J. Math. Control Inf.*, 33(2):257–291, 2016.
- [PTW16b] R. Picard, S. Trostorff, and M. Waurick. On maximal regularity for a class of evolutionary equations. *J. Math. Anal. Appl.*, 2016. <http://dx.doi.org/10.1016/j.jmaa.2016.12.057>.

Bibliography

- [PTWW13] R. Picard, S. Trostorff, M. Waurick, and M. Wehowski. On Non-autonomous Evolutionary Problems. *J. Evol. Equ.*, 13(4):751–776, 2013.
- [PW34] R. E. Paley and N. Wiener. Fourier transforms in the complex domain. (Am. Math. Soc. Colloq. Publ. 19) New York: Am. Math. Soc. VIII, 1934.
- [QR07] R. Quintanilla and R. Racke. Qualitative aspects in dual-phase-lag heat conduction. *Proc. R. Soc. Lond., Ser. A, Math. Phys. Eng. Sci.*, 463(2079):659–674, 2007.
- [QR08] R. Quintanilla and R. Racke. A note on stability in three-phase-lag heat conduction. *Int. J. Heat Mass Transfer*, 51(1-2):24–29, 2008.
- [Qui02] R. Quintanilla. Exponential stability in the dual-phase-lag heat conduction theory. *J. Non-Equilibrium Thermodyn.*, 27(3):217–227, 2002.
- [Rad19] H. Rademacher. Über partielle und totale Differenzierbarkeit von Funktionen mehrerer Variablen I. *Math. Ann.*, 79:340–359, 1919.
- [Rei07] T. Reis. Consistent initialization and perturbation analysis for abstract differential-algebraic equations. *Math. Control Signals Syst.*, 19(3):255–281, 2007.
- [Rud87] W. Rudin. *Real and complex analysis*. Mathematics series. McGraw-Hill, 1987.
- [Rud91] W. Rudin. *Functional analysis*. 2nd ed. New York, NY: McGraw-Hill, 2nd ed. edition, 1991.
- [Rue09] W. M. Ruess. Flow invariance for nonlinear partial differential delay equations. *Trans. Am. Math. Soc.*, 361(8):4367–4403, 2009.
- [Sae70] R. Sacks. Causality in Hilbert space. *SIAM Rev.*, 12:357–383, 1970.
- [Sho97] R. E. Showalter. *Monotone Operators in Banach Space and Nonlinear Partial Differential Equations*. American Mathematical Society, 1997.
- [Sho00] R. Showalter. Diffusion in poro-elastic media. *J. Math. Anal. Appl.*, 251(1):310–340, 2000.
- [SS03] E. Stein and R. Shakarchi. *Fourier Analysis: An Introduction*. Number Bd. 10 in Princeton Lectures in Analysis. Princeton University Press, 2003.
- [Tro11] S. Trostorff. *Well-posedness and causality for a class of evolutionary inclusions*. PhD thesis, TU Dresden, 2011.
- [Tro12] S. Trostorff. An alternative approach to well-posedness of a class of differential inclusions in Hilbert spaces. *Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods*, 75(15):5851–5865, 2012.
- [Tro13a] S. Trostorff. Autonomous Evolutionary Inclusions with Applications to Problems with Nonlinear Boundary Conditions. *Int. J. Pure Appl. Math.*, 85(2):303–338, 2013.
- [Tro13b] S. Trostorff. Exponential Stability for Linear Evolutionary Equations. *Asymptotic Anal.*, 85:179–197, 2013.
- [Tro14a] S. Trostorff. A characterization of boundary conditions yielding maximal monotone operators. *J. Funct. Anal.*, 267(8):2787–2822, 2014.

Bibliography

- [Tro14b] S. Trostorff. A Note on Exponential Stability for Evolutionary Equations. *Proc. Appl. Math. Mech.*, 14:983–984, 2014.
- [Tro15a] S. Trostorff. Exponential stability for second order evolutionary problems. *J. Math. Anal. Appl.*, 429(2):1007–1032, 2015.
- [Tro15b] S. Trostorff. On a class of block operator matrices in system theory. Technical report, TU Dresden, 2015. arXiv:1502.05496.
- [Tro15c] S. Trostorff. On Integro-Differential Inclusions with Operator-valued Kernels. *Math. Methods Appl. Sci.*, 38(5):834–850, 2015.
- [TW14] S. Trostorff and M. Wehowski. Well-posedness of non-autonomous evolutionary inclusions. *Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods*, 101:47–65, 2014.
- [Tzo95] D. Tzou. A unified field approach for heat conduction from macro-to micro-scales. *J. Heat Transfer*, 117(1):8–16, 1995.
- [Wal03] H.-O. Walther. Differentiable semiflows for differential equations with state-dependent delays. *Zesz. Nauk. Uniw. Jagiell., Univ. Jagell. Acta Math.*, 1269:57–66, 2003.
- [Wal09] H.-O. Walther. Algebraic-delay differential systems, state-dependent delay, and temporal order of reactions. *J. Dyn. Differ. Equations*, 21(1):195–232, 2009.
- [Wau12] M. Waurick. A Hilbert space approach to homogenization of linear ordinary differential equations including delay and memory terms. *Math. Methods Appl. Sci.*, 35(9):1067–1077, 2012.
- [Wau13] M. Waurick. Homogenization of a class of linear partial differential equations. *Asymptotic Anal.*, 82(3-4):271–294, 2013.
- [Wau14a] M. Waurick. G -convergence of linear differential equations. *Z. Anal. Anwend.*, 33(4):385–415, 2014.
- [Wau14b] M. Waurick. Homogenization in fractional elasticity. *SIAM J. Math. Anal.*, 46(2):1551–1576, 2014.
- [Wau15a] M. Waurick. A note on causality in reflexive Banach spaces. *Indag. Math.*, 26(2):404–412, 2015.
- [Wau15b] M. Waurick. On non-autonomous integro-differential-algebraic evolutionary problems. *Math. Methods Appl. Sci.*, 38(4):665–676, 2015.
- [Wau16a] M. Waurick. *On the continuous dependence on the coefficients of evolutionary equations*. Habilitation thesis, TU Dresden, 2016.
- [Wau16b] M. Waurick. On the homogenization of partial integro-differential-algebraic equations. *Oper. Matrices*, 10(2):247–283, 2016.
- [Web76] G. Webb. Functional differential equations and nonlinear semigroups in L^p spaces. *J. Differ. Equations*, 20:71–89, 1976.
- [Weh15] M. Wehowski. *Well-posedness of degenerate nonlinear Cauchy problems in Hilbert spaces*. PhD thesis, TU Dresden, 2015.

Bibliography

- [Wei80] J. Weidmann. *Linear Operators in Hilbert Spaces*. Springer, 1980.
- [Wei91] G. Weiss. Representation of shift-invariant operators on L^2 by H^∞ transfer functions: An elementary proof, a generalization to L^p , and a counterexample for L^∞ . *Math. Control Signals Syst.*, 4(2):193–203, 1991.
- [Wei95] L. Weis. The stability of positive semigroups on L_p spaces. *Proc. Am. Math. Soc.*, 123(10):3089–3094, 1995.
- [Wei01] L. Weis. Operator-valued Fourier multiplier theorems and maximal L_p -regularity. *Math. Ann.*, 319(4):735–758, 2001.
- [Wid34] D. Widder. The inversion of the Laplace integral and the related moment problem. *Trans. Am. Math. Soc.*, 36:107–200, 1934.
- [Wid71] D. Widder. An introduction to transform theory. Pure and Applied Mathematics, 42. New York-London: Academic Press. XIV, 1971.
- [Yos95] K. Yosida. *Functional Analysis*. Springer, 6th edition, 1995.

Erklärung

Hiermit versichere ich, dass ich die vorliegende Arbeit ohne unzulässige Hilfe Dritter und ohne Benutzung anderer als der angegebenen Hilfsmittel angefertigt habe; die aus fremden Quellen direkt oder indirekt übernommenen Gedanken sind als solche gekennzeichnet. Die Arbeit wurde bisher weder im Inland noch im Ausland in gleicher oder ähnlicher Form einer anderen Prüfungsbehörde vorgelegt.

Es wurden zuvor keine Habilitationsvorhaben unternommen.

Ich erkenne die Habilitationsordnung der Fakultät Mathematik und Naturwissenschaften der TU Dresden vom 12.12.2010 an.

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